Optimality properties for the estimation of jumps in stochastic processes.

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Intro : Estimation of jumps in a stochastic process

Consider a diffusion with jumps $X = (X_t)_{t\in[0,1]}$,

$$X_t = X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t \int_{\mathbf{E}} \kappa \circ c(s,e)(\mu - \nu)(\mathrm{d}s, \mathrm{d}e) + \int_0^t \int_{\mathbf{E}} \kappa' \circ c(s,e)\mu(\mathrm{d}s, \mathrm{d}e)$$

B is a standard Brownian motion
b and σ are adapted processes
μ is a random poisson measure with auxiliary space $\mathbf{E}$,
ν the intensity measure on $[0,1] \times \mathbf{E}$
$c(s,e) = c_\omega(s,e)$ predictable function, $\kappa(x) = x$ on neighbourhood of zero, $\kappa'(x) = 1 - x$.

Statistical problems

One observe discretely the path of X : $(X_{i/n})_{i=0,\dots,n}$.

Interesting estimation problems (already studied in several references) :

1. Estimation of volatility $[X,X]_1 = \int_0^1 \sigma_s^2 ds + \sum_{s\leq 1} \Delta X_s^2$
2. Estimation of some functional $\sum_{s\leq 1} H(\Delta X_s)$, $H(x) = x^2$, $H(x) = |x|^3$, $H(x) = |x|^p$...
3. Are there jumps in the model (in the observed path) ?
4. Do different components jump together ?
5. What is the degree of intensity of the jump component ?

Non parametric estimation of the (random) realization of the jump component
For instance connected to modelisation of asset prices.

Our question : Optimality in the problem 2)

Some references about non parametric estimation of jumps components

- *Mancini (2001), (2009)* Threshold methods
- *Barndorff-Nielsen & Shephard (2006)* Multipower variation, estimation of volatility
- *Barndorff-Nielsen, Graversen, Jacod, Podolskij, Shephard (2006)*
- *Jacod (2008)* Estimation of $\sum_{s\leq 1} H(\Delta X_s)$
- *Aït-Sahalia & Jacod (2009)* Testing for jumps
- *Jacod & Todorov (2009)* Testing for common jumps
- *Aït-Sahalia & Jacod (2009)* Estimating degree of jump
- *Shimizu (07)* Non parametric estimation of the compensator
- *Neuman & Reiß(09)* Non parametric estimation of the compensator

Estimation of functionals of the jumps *Jacod 08*

$$X_t = X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t \int_{\mathbf{E}} \kappa \circ c(s,e)(\mu - \nu)(\mathrm{d}s, \mathrm{d}e) + \int_0^t \int_{\mathbf{E}} \kappa' \circ c(s,e)\mu(\mathrm{d}s, \mathrm{d}e)$$

Define $V_n(H) = \sum_{i=0}^{n-1} H(X_{\frac{i+1}{n}} - X_{\frac{i}{n}})$ and $V(H) = \sum_{s \leq 1} H(\Delta X_s)$.

Theorem

Assume some regularity on the coefficients, $\underline{\nu}(dt, de) = dt de$*,* $s \mapsto \int_{\mathbf{E}} \kappa \circ c_\omega(s,e)^2 de$ *is a predictible locally bounded process. Then,*

$$V_n(H) \xrightarrow{n \to \infty} V(H), \text{ if } H(x) = o(x^2) \text{ near } 0$$

$$V_n(H) \xrightarrow{n \to \infty} \int_0^1 \sigma_s^2 ds + \sum_{s \leq 1} (\Delta X_s)^2 \text{ for } H(x) = x^2$$

Estimation of functionals of the jumps *Jacod 08*

Denote $(T_p)_{p\geq 1}$ the jump times of X, $V(H) = \sum_p H(\Delta X_{T_p})$.

Theorem (Jacod (08))

If H is $\mathcal{C}^2$, $H(0) = H'(0) = 0$, $H''(x) = o(x)$ near 0

$$\sqrt{n}(V_n(H) - V(H)) \xrightarrow{n\to\infty} Z(H') \text{ stably in law}$$

where $Z(H')$ is a conditionally Gaussian variable

$$Z(H') = \sum_p H'(\Delta X_{T_p})[\sqrt{U_p} Z_p^{(1)} \sigma_{T_p -} + \sqrt{(1-U_p)} Z_p^{(2)} \sigma_{T_p}]$$

where $(U_p)_p$ is a i.i.d. sequence of uniform variables on $[0,1]$, $(Z_p^{(1)})_p$, $(Z_p^{(2)})_p$ are i.i.d. sequences of $\mathcal{N}(0,1)$ variables

Statistics for test of presence of jumps are based on $V_n(H)$ at different scale.

Optimality questions

- Optimality of these test procedures ?
- $V_n(H)$ optimal for estimating the functional of the jumps ?
- Is the error term $\sum_p H'(\Delta X_{T_p})[\sigma_{T_p-}\sqrt{U_p}Z_p^{(1)} + \sigma_{T_p}\sqrt{(1-U_p)}Z_p^{(2)}]$ the minimal one ?
- What is optimal for estimating ΔX_{T_p} ?
- Which mathematical meaning should be given for the optimality ?

For simplicity : focus on the estimation of the jumps.

Optimal estimation of the jumps of the process

Consider a diffusion with a finite number of jumps :

$$X_t = X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t \int_{\mathbf{E}} c(s,e)\mu(\mathrm{d}s, \mathrm{d}e)$$

B is a standard Brownian motion
b and σ are adapted processes with c.a.d.l.a.g. paths, c c.a.g.l.a.d.
μ is a random poisson measure with intensity $\nu = \mathrm{d}s \times \mathrm{d}\lambda$ and

$$\lambda(E) < \infty$$

Observations : $(X_{\frac{i}{n}}) \quad i = 0, \ldots n.$

Estimation results

$$X_t = X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t \int_{\mathbf{E}} c(s,e)\mu(\mathrm{d}s, \mathrm{d}e)$$

Denote K the number of jumps for X and $0 \leq T_1 < \cdots < T_K \leq 1$ the jumps instants.

Prop

There exist $\widehat{K}$ *and* $\widehat{\Delta X} = (\widehat{\Delta X}_1, \ldots, \widehat{\Delta X}_{\widehat{K}}, 0, 0, \ldots)$ *estimators such that :*

$$P(\widehat{K} \neq K) \xrightarrow{n \to \infty} 0,$$

$$(\widehat{\Delta X}_1, \ldots, \widehat{\Delta X}_K) \xrightarrow{n \to \infty} (\Delta X_{T_1}, \ldots, \Delta X_{T_K}) \textit{in probability}$$

Prop (associated C.L.T)

We have $\sqrt{n}(\widehat{\Delta X}_j - \Delta X_{T_j})_{j \le K}$ *converges stably in law to*

$$(\sigma_{T_j-}\sqrt{U_j}Z_j^{(1)} + \sigma_{T_j}\sqrt{1-U_j}Z_j^{(2)})_{j \le K}$$

where $(Z_j^{(1)})_{j\ge 1}$, $(Z_j^{(2)})_{j\ge 1}$, *are i.i.d.* $\mathcal{N}(0,1)$, $(U_j)_{j\ge 1}$ *is i.i.d. uniform on* $[0,1]$.

More precisely,

$$E[f(\sqrt{n}(\widehat{\Delta X}_j - \Delta X_{T_j})_{j\le k})1_{K=k}\Psi] \xrightarrow{n\to\infty} E[f((\sigma_{T_j-}\sqrt{U_j}Z_j^{(1)} + \sigma_{T_j}\sqrt{1-U_j}Z_j^{(2)})_{j\le k})1_{K=k}\Psi]$$

for any bounded continuous $f : \mathbb{R}^k \to \mathbb{R}$, $k \ge 1$, Ψ *random variable* X*-measurable.*

Rk : The limit law is conditionally Gaussian $\mathcal{N}(0,V)$ with diagonal variance matrix, $V_{jj} = \sigma^2_{T_j-}U_j + \sigma^2_{T_j}(1-U_j), \quad j \le k.$

Constuction of the estimator

1. Determine increments greater than typical brownian increments. For $\omega \in (0, 1/2)$

$$\hat{i}_1 = \inf\{1 \leq i \leq n \mid \left|X_{\frac{i}{n}} - X_{\frac{i-1}{n}}\right| > n^{-\omega}\} \quad (= \infty \text{ if empty})$$

$$\vdots$$

$$\hat{i}_j = \inf\{\hat{i}_{j-1} < i \leq n \mid \left|X_{\frac{i}{n}} - X_{\frac{i-1}{n}}\right| > n^{-\omega}\} \quad (= \infty \text{ if empty})$$

   etc . . . .
   And set $\widehat{K} = \sup\{j \mid \hat{i}_j < \infty\}$.

2. Estimate the jump by the increment. We set

$$\widehat{\Delta X}_1 = X_{\frac{\hat{i}_1}{n}} - X_{\frac{\hat{i}_1 - 1}{n}} \quad (= 0 \text{ if } \hat{i}_1 = \infty)$$

$$\vdots$$

$$\widehat{\Delta X}_j = X_{\frac{\hat{i}_j}{n}} - X_{\frac{\hat{i}_j - 1}{n}} \quad (= 0 \text{ if } \hat{i}_j = \infty)$$

3. Finally, $\widehat{\Delta X} = (\widehat{\Delta X}_1, \ldots, \widehat{\Delta X}_{\widehat{K}}, 0, 0, \ldots)$

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'Explanation' for the CLT

Let $\frac{i_p}{n} \leq T_p < \frac{i_p+1}{n}$.

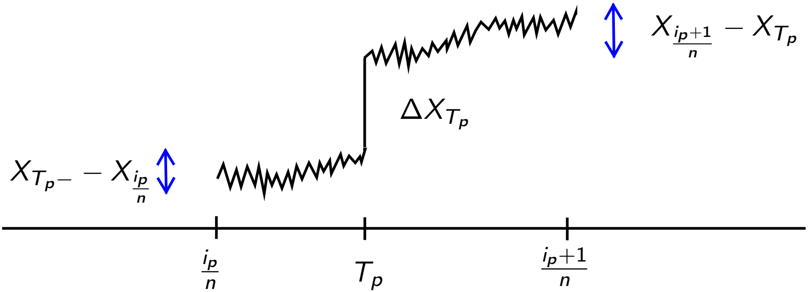
Error between an exact jump and corresponding increment

$$\sqrt{n}[(X_{\frac{i_p+1}{n}} - X_{\frac{i_p}{n}}) - \Delta X_{T_p}] \xrightarrow{n\to\infty} \sigma_{T_p -}\sqrt{U_p} Z_p^{(1)} + \sigma_{T_p}\sqrt{(1-U_p)} Z_p^{(2)}$$

Using results by *Jacod (08)*

$$\sqrt{n}(X_{T_p -} - X_{\frac{i_p}{n}}) \simeq \sigma_{T_p -}\sqrt{n}(B_{T_p} - B_{\frac{i_p}{n}}) \to \sigma_{T_p -}\sqrt{U_p} Z_p^{(1)}$$

$$\sqrt{n}(X_{\frac{i_p+1}{n}} - X_{T_p}) \simeq \sigma_{T_p}\sqrt{n}(B_{\frac{i_p+1}{n}} - B_{T_p}) \to \sigma_{T_p}\sqrt{(1-U_p)} Z_p^{(2)}$$



Optimality result

Assumptions on the model :

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}B_s + \int_0^t \int_{\mathbb{R}} c(X_{s-}, e)\mu(\mathrm{d}s, \mathrm{d}e)$$

1. σ, b, c are smooth coefficients, bounded + bounded derivatives
2. μ is a random measure on $[0,1] \times \mathbb{R}$ independent of $(B_t)_t$
   For simplicity the number of jumps $k \geq 1$ is fixed :

$$\mu = \sum_{j=1}^{k} \delta_{(T_j, \Lambda_j)}$$

with $T_1 < \cdots < T_k$ jump times, $(\Lambda_1, \ldots, \Lambda_k) \in \mathbb{R}^k$.

- $(T_1, \ldots, T_k)$ has a density on $[0,1]^k$
- $(\Lambda_1, \ldots, \Lambda_k)$ has a density on $\mathbb{R}^k$

Remark : Markovian dependence of σ, b, c

Theorem

Suppose that σ^{-1}, $[\frac{\partial c(x,e)}{\partial e})]^{-1}$, $[1+\frac{\partial c(x,e)}{\partial x}]^{-1}$ *are bounded.*
Assume that $U_n = f_n(X_{i/n}, i = 0, \ldots, n)$ *is a sequence of estimators with values in* $\mathbb{R}^k$ *such that*

$$\sqrt{n}\left(\underbrace{U_n}_{estimator} - \underbrace{\Delta X}_{true\ jumps}\right) \xrightarrow[law]{n\to\infty} Z$$

where $\Delta X = (\Delta X_{T_1}, \ldots, \Delta X_{T_k}) = (c(X_{T_1-}, \Lambda_1), \ldots, c(X_{T_k-}, \Lambda_k))$ *and* Z *is some law on* $\mathbb{R}^k$.
Then, Z *admits a convolution structure :*

$$Z \stackrel{law}{=} I^{-1/2}N + W,$$

where :

- I *is a diagonal random matrix (information)*
- N *gaussian* $\mathcal{N}(0, Id_k)$
- W *is independent of* N *conditionally to* I.

The random information matrix is

$$I = \text{diag}(I_1, \ldots, I_k)$$

with

$$I_j = [\sigma^2(X_{T_j}-)U_j + \sigma^2(X_{T_j})(1 - U_j)]^{-1}$$

where $(U_1, \ldots, U_k)$ are i.i.d. uniform on $[0, 1]$.

Consequence of $Z \overset{\text{law}}{=} I^{-1/2}N + W$

1. At best, the jumps are estimated with conditionally independent Gaussian errors (with random variances $(I_j^{-1})_j$).
2. The estimator $\widehat{\Delta X}$ has the law with the minimal dispersion $(W = 0)$.

Preliminary study : LAMN for a parametric model associated to our problem

- The initial model

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s + \sum_{T_j \le t} c(X_{T_j -}, \Lambda_j)$$

- Parametric model : For $\theta \in \mathbb{R}^k$, define :

$$X_t^\theta = X_0^\theta + \int_0^t b(X_s^\theta)\mathrm{d}s + \int_0^t \sigma(X_s^\theta)\mathrm{d}W_s + \sum_{T_j \le t} c(X_{T_j -}^\theta, \theta_j)$$

- Denote $\mathbf{p}^{n,\theta}$ the law of the observation

$$\mathrm{Obs} = \{X_{i/n}^\theta, \quad i = 1, \dots n; T_1, \dots T_k\}$$

Rk : The parameter θ is not the jumps of X (unless $c(x, e) = e$)

Theorem

The statistical model $(\mathbf{p}^{n,\theta})_{\theta\in\mathbb{R}^k}$ *satisfies a LAMN property. For* $\theta \in \mathbb{R}^k$, $h \in \mathbb{R}^k$ *:*

$$\ln \frac{\mathbf{p}^{n,\theta+h/\sqrt{n}}}{\mathbf{p}^{n,\theta}}(Obs) = \sum_{j=1}^{k} h_j \widetilde{I}_n(\theta)_j^{1/2} \widetilde{N_n}(\theta)_j - \frac{1}{2}\sum_{j=1}^{k} h_j^2 \widetilde{I}_n(\theta)_j + o_{\mathbf{p}^{n,\theta}}(1)$$

where $(\widetilde{I}_n(\theta), \widetilde{N_n}(\theta)) \xrightarrow[law]{n\to\infty} (\widetilde{I}(\theta), \widetilde{N})$ *with :*

$$\widetilde{I}(\theta)_j = \frac{\dot{c}(X_{T_j-},\theta)^2}{\sigma^2(X_{T_j-})[1 + c'(X_{T_j-},\theta)]^2 U_j + \sigma^2(X_{T_j})(1-U_j)}$$

$$\widetilde{N} \sim \mathcal{N}(0, Id_k)$$

Comment on the information

- $\dot{c}(X_{T_j-},\theta)$ is not surprising
- $[1+c'(X_{T_j-},\theta)]^2$ comes from the fact that the value of the jump is $c(X_{T_p-},\theta)$ and not $c(X_{\frac{i_p}{n}},\theta)$ where $\frac{i_p}{n} \leq T_p < \frac{i_p+1}{n}$.
- If $c(x,e)=e$ then $\widetilde{I}=I$

Expression for the likelihood of the model

$$\mathbf{p}^{n,\theta}(\text{Obs}) = \underbrace{f(T_1,\ldots,T_k)}_{\text{density of jump times}} \times \underbrace{f_\theta(X_{i/n}, i=0,\ldots,n \mid T_1,\ldots,T_k)}_{\text{density of the diffusion cond. to jump times}}$$

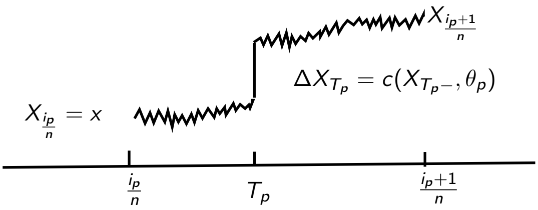
- Conditionally to jump times, $(X_{i/n})_i$ is Markov inhomogeneous.
- The transition of $X_{i/n}$ to $X_{(i+1)/n}$:
  - does not depend on θ if there is no jump in $[\frac{i}{n}, \frac{i+1}{n}]$
  - depends on θ_p if $T_p \in [\frac{i}{n}, \frac{i+1}{n}]$

$$\frac{\mathbf{p}^{n,\theta+h/\sqrt{n}}}{\mathbf{p}^{n,\theta}}(\mathrm{Obs}) = \frac{f(T_1,\dots,T_k)\prod_{i=0}^{n-1} p^{\theta+h/\sqrt{n}}_{\frac{i}{n},\frac{i+1}{n}}(X_{\frac{i}{n}},X_{\frac{i+1}{n}})}{f(T_1,\dots,T_k)\prod_{i=0}^{n-1} p^{\theta}_{\frac{i}{n},\frac{i+1}{n}}(X_{\frac{i}{n}},X_{\frac{i+1}{n}})}$$

$$= \prod_{p=1}^{k} \frac{p^{\theta_p+h_p/\sqrt{n}}_{\frac{i_p}{n},\frac{i_p+1}{n}}(X_{\frac{i_p}{n}},X_{\frac{i_p+1}{n}})}{p^{\theta_p}_{\frac{i_p}{n},\frac{i_p+1}{n}}(X_{\frac{i_p}{n}},X_{\frac{i_p+1}{n}})}$$

with $\frac{i_p}{n} \leq T_p < \frac{i_p+1}{n}$.

$p^{\theta_p}_{\frac{i_p}{n},\frac{i_p+1}{n}}(x,y)$
transition of :



Theorem (Expansion of the score function, for the transition with one jump)

We have :

$$\frac{\dot{p}^{\theta_p}_{\frac{i_p}{n},\frac{i_p+1}{n}}}{p^{\theta_p}_{\frac{i_p}{n},\frac{i_p+1}{n}}}(x,y) =$$

$$\frac{\dot{c}(x,\theta_p)n[y-x-c(x,\theta_p)]}{\sigma^2(x)[1+c'(x,\theta_p)]^2[nT_p-i_p]+\sigma^2(x+c(x,\theta_p))[i_p+1-nT_p]}+r_n(x,y)$$

Asymptotic expansions using Malliavin calculus. *Gobet 01, Gobet 02, G. Gobet 08*

To understand :

$$\mathrm{Law}(X_{\frac{i_p+1}{n}} - X_{\frac{i_p}{n}} \mid X_{\frac{i_p}{n}} = x) \simeq \sigma(x)[B_{T_p} - B_{\frac{i_p}{n}}] + \underbrace{c(x+\sigma(x)[B_{T_p} - B_{\frac{i_p}{n}}],\theta_p)}_{\text{approximate jump}} + \sigma(x+c(x,\theta_p))[B_{\frac{i_p+1}{n}} - B_{T_p}]$$

.

We deduce...

$$\ln \frac{\mathbf{p}^{n,\theta+h/\sqrt{n}}}{\mathbf{p}^{n,\theta}}(\mathrm{Obs}) = \sum_{p=1}^{k} h_p \sqrt{n}\dot{c}(X_{\frac{i_p}{n}},\theta)\left(X_{\frac{i_{p+1}}{n}} - X_{\frac{i_p}{n}} - c(\frac{i_p}{n},\theta)\right) -$$

$$\frac{1}{2}\sum_{p=1}^{k} \frac{h_p^2 \dot{c}(X_{\frac{i_p}{n}})^2}{\left[\sigma^2(X_{\frac{i_p}{n}})[1+c'(X_{\frac{i_p}{n}},\theta)]^2(nT_p - i_p) + \sigma^2(X_{\frac{i_p}{n}} + c(X_{\frac{i_p}{n}},\theta))(i_{p+1} - nT_p)\right]}$$

$$+ o(1)$$

and then LAMN.

From the parametric problem to the estimation of random jumps

Theorem (Hajek's convolution Theorem *Jeganathan (82)*)

Let $(\mathbf{p}^{n,\theta})_\theta$ satisfies LAMN with info $\widetilde{I}(\theta)$.
*If $(\widetilde{\theta}_n)$ is a '**regular**' estimator :*
$$\forall h, \quad \sqrt{n}[\widetilde{\theta}_n - (\theta + h/\sqrt{n})] \xrightarrow{n\to\infty} Z \quad \text{in law under } \mathbf{p}^{n,\theta+\frac{h}{\sqrt{n}}}$$
Then, Z has a convolution structure : $Z = \widetilde{I}(\theta)^{-1/2}N + W$, N standard Gaussian, W independent of N conditional to $I(\theta)$.

- Our result on the estimation of the random jumps :

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s + \sum_{T_j \le t} \underbrace{c(X_{T_j-}, \Lambda_j)}_{\Delta X_{T_j}}$$

Theorem

*If (U_n) is **any** estimator such that $\sqrt{n}(U_n - \Delta X) \xrightarrow[law]{n\to\infty} Z$. Then, Z admits a convolution structure : $Z \stackrel{law}{=} I^{-1/2}N + W$*

Idea :

- Equation $\sqrt{n}(U_n - c(X_{T_p}, \Lambda_p)_p) \to Z$ acts as a regularity assumption :
  For simplicity take $c(x, e) = e$

$$\sqrt{n}\,(U_n - \Lambda) \xrightarrow[law]{n\to\infty} Z$$

  Λ admits a density : Laws of Λ and $\Lambda + \frac{h}{\sqrt{n}}$ are close.
  $\to$ U_n has to be 'regular' and convolution theorem is shown with methods similar to *Jeganathan (82)*
- If c depends on X. 'Convolution' theorem is more difficult : $\Delta X_{T_p} = c(X_{T_p}, \Lambda_p)$ depends on 'parameter' Λ_p and the unobserved X_{T_p}.

Sketch of the proof in the case $c(x,e)=e$

$$
\begin{aligned}
&E[f(\sqrt{n}(U_n-\Delta X))]\\
&=\int_{\mathbb{R}^k} E^{n,\theta}[f(\sqrt{n}(U_n-\theta))]f_\Lambda(\theta)\mathrm{d}\theta\\
&=\int_{\mathbb{R}^k} E^{n,\theta+\frac{h}{\sqrt{n}}}\left[f(\sqrt{n}(U_n-\theta-\frac{h}{\sqrt{n}}))\right]f_\Lambda(\theta+\frac{h}{\sqrt{n}})\mathrm{d}\theta\\
&=\int_{\mathbb{R}^k} E^{n,\theta+\frac{h}{\sqrt{n}}}\left[f(\sqrt{n}(U_n-\theta-\frac{h}{\sqrt{n}}))\right]f_\Lambda(\theta)\mathrm{d}\theta+o(1)\\
&=\int_{\mathbb{R}^k} E^{n,\theta}[f(\sqrt{n}(U_n-\theta-\frac{h}{\sqrt{n}}))\frac{\mathrm{d}\mathbf{p}^{n,\theta+\frac{h}{\sqrt{n}}}}{\mathrm{d}\mathbf{p}^{n,\theta}}]f_\Lambda(\theta)\mathrm{d}\theta+o(1)\\
&=E[f(\sqrt{n}(U_n-\theta-\frac{h}{\sqrt{n}}))\frac{\mathrm{d}\mathbf{p}^{n,\theta+\frac{h}{\sqrt{n}}}}{\mathrm{d}\mathbf{p}^{n,\theta}}]+o(1)
\end{aligned}
$$

Taking limits :

$$E[f(Z)]=E(f(Z-h)e^{h'I^{1/2}N-h'Ih})\quad \text{for all } h$$

"Characterisation of the law of Z"