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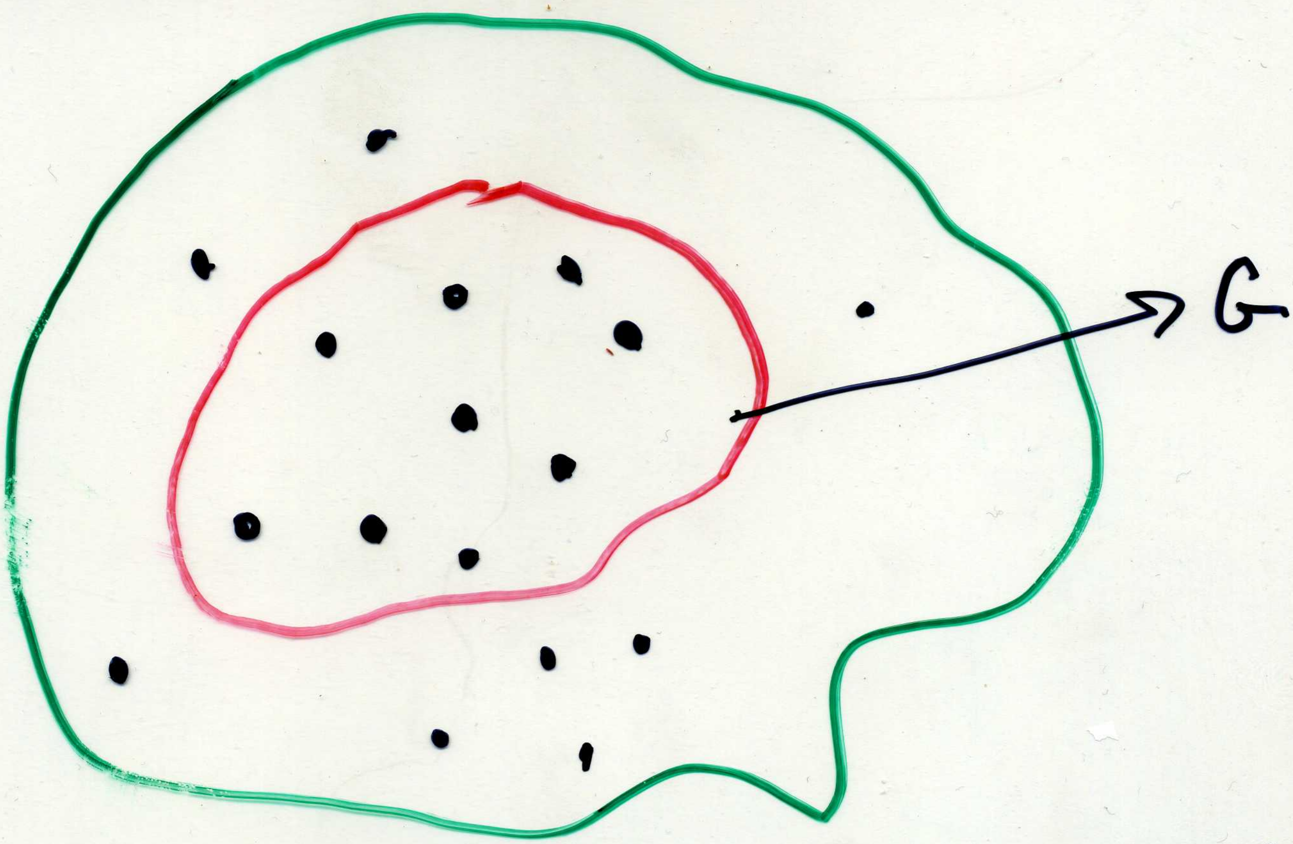
On the estimation of the intensity density of Poisson random fields

I. The poblem

1. Observations. $\Pi_\varepsilon \subseteq G \subseteq R^d$



the intensity density $\lambda_\varepsilon(x) = \dfrac{\lambda(x)}{\varepsilon}$

The problem: to estimate λ,

asymptotic: $\varepsilon \to 0$

Estimates:

I. $$\hat{\lambda}_\varepsilon(x) = \sum_{X_i \in \Pi_\varepsilon} K_\varepsilon(x; X_i)$$

1. $$\hat{\lambda}_\varepsilon = \sum a_\varepsilon K(a_\varepsilon(x - X_i))$$

2. $$\hat{\lambda}_\varepsilon = \sum_{\nu=1}^{N(\varepsilon)} \varphi_\nu(x) \Big[\varepsilon \sum_{\Pi_\varepsilon} \varphi_\nu(X_i) \Big]$$

Example

$\Pi_1 \dots \Pi_n$

$$\Pi_\varepsilon = \cup \Pi_n, \quad \lambda_\varepsilon = n\lambda.$$

Apriori data:

$$\lambda \in \Lambda.$$

II. General classes Λ

1. Λ is a metric space of functions on G.

$H_\delta(\Lambda)$, $C_\delta(\Lambda)$ – δ-entropy and δ- capacity of Λ

Result II.1 Consist. est. exist. iff $\Leftrightarrow$ $H_\delta < \infty$

Result II.2

$$E_\lambda \|\lambda - \hat{\lambda}_\varepsilon\|_2^2 \le \inf_\delta \left(\delta^2 + \exp\left\{C_\delta + a \cdot \frac{\delta^2}{\varepsilon}\right.\right.$$

2. Result II.3

$$E_\lambda \|\lambda - \hat{\lambda}_\varepsilon\|_2 \le \inf_N \left(\|\lambda\|_\infty \cdot N \cdot \varepsilon + 2 d_N^2 \right)$$

3. Result II.4. Let $\delta > 0$

$\exists\ \lambda_{i\delta} \in \Lambda$, $\rho(\lambda_{i\delta}, \lambda_{j\delta}) > \delta$, $i = 1, 2, \dots N(\delta)$. Set

$$S(\varepsilon, \Lambda) = \sup_{\lambda_0, \{\lambda_{\varepsilon\delta}} \Big\{ \delta : [\ln N(\delta)]^{-1} \max_{1 \le i \le N(\delta)} \left\| \frac{\lambda_{i\varepsilon} - \lambda_0}{\sqrt{\lambda_0}} \right\|_{L_2(\mu)} \le \frac{\varepsilon}{2} \Big\}.$$

Then

$$\sup_\lambda E_\lambda \, l\left(\frac{\rho(\hat{\lambda}_\varepsilon; \lambda)}{S(\varepsilon; \Lambda)} \right) \ge \frac{1}{2} \, l\left(\frac{1}{2}\right).$$

III Smooth functions

$$\lambda \in L_p(G), \quad G \subset R^d$$

$$\lambda = \lambda(x_1, x_2, \dots x_d)$$

$$\beta_i = k_i + \alpha_i$$

$$\beta : \quad \frac{1}{\beta} = \frac{1}{\beta_1} + \dots + \frac{1}{\beta_d}$$

Result III.1

$$\inf_{\tilde{\lambda}_\varepsilon} \sup_{\lambda \in \Lambda} E_\lambda \| \tilde{\lambda}_\varepsilon - \lambda \|_p \asymp$$

$$\asymp \begin{cases} \varepsilon^{\frac{\beta}{2\beta+1}}, & 1 \le p \le 2 \\ \varepsilon^{\frac{\beta}{2\beta+1}}, & 2 \le p < \infty \\ \left(\varepsilon \ln \frac{1}{\varepsilon}\right)^{\frac{\beta}{2\beta+1}} & \end{cases}$$

IV Analytic functions

IV.1 The class $\mathcal{A}(M, G)$

$$G \subset \mathbb{C}^d$$

$\lambda(x_1, \dots x_d)$ on $F \subset \mathbb{R}^d$

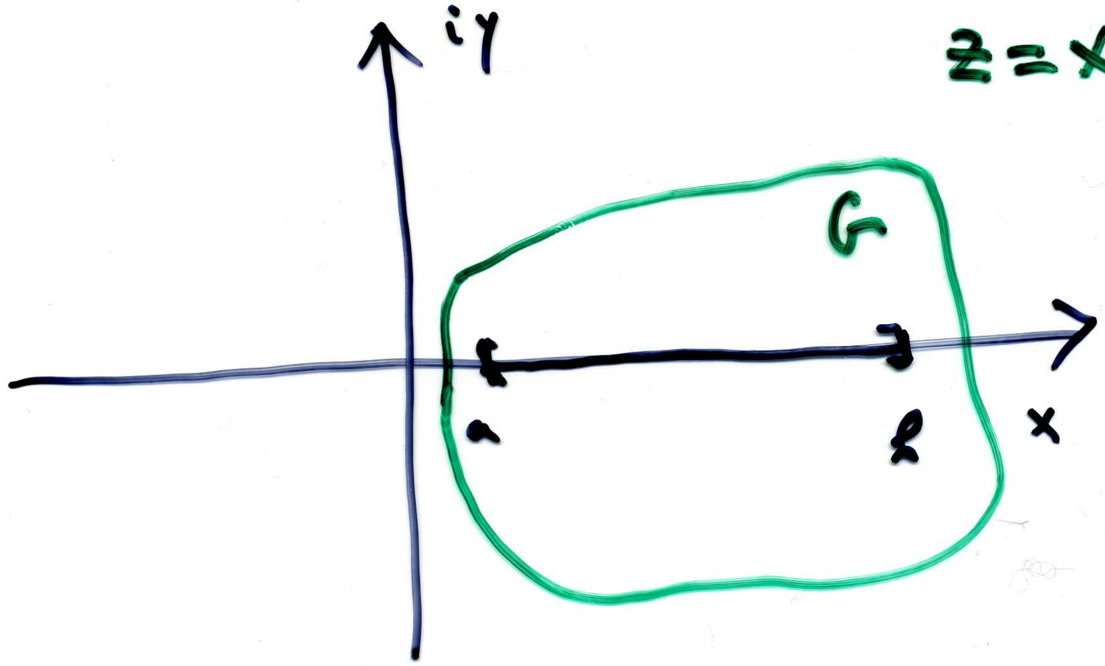
$F \subset G$, $\bar{\lambda}(z_1, \dots z_d)$ on G

$\bar{\lambda} = \lambda$ on F.

Example: $d = 1$, $F = [a, b]$

$$[a, b] \subset G$$

$z = x + iy$



Result IV.1. $\lambda \in \mathcal{A}(M,G)$

$$\Delta_p(\varepsilon) = \inf_{\hat\lambda_\varepsilon} \sup_{\lambda} E_\lambda \|\lambda - \hat\lambda_\varepsilon\|_{L_p(}$$

Then

$$\Delta_p(\varepsilon) \asymp \left(\varepsilon\, |\ln \varepsilon|^{-d}\right)^{1/2}, \quad p<4$$

$$\Delta_p(\varepsilon) \asymp \left(\varepsilon |\ln \varepsilon|^{-d}\right)^{1/2} \left(\ln \ln \frac{1}{\varepsilon}\right)^{\frac{d}{4}}$$

$\cancel{\Delta_\infty(\varepsilon) \asymp}$

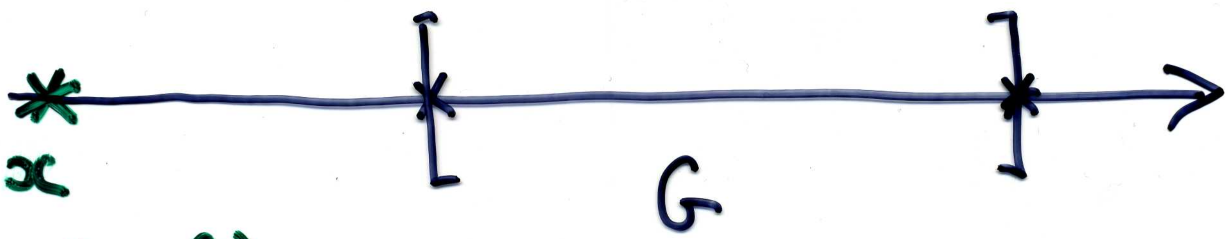
$$\Delta_p(\varepsilon) \asymp \left(\varepsilon |\ln \varepsilon|^{-d}\right)^{1/2} |\ln \varepsilon|^{d(1-\frac{2}{p}}$$

The rate is almost „parametric".

V Extrapolation of analytic densities.

1. The problem. We observe Π_ε in G and estimate $\lambda(x)$ at ~~t~~ a point $x \notin G$. Is it possible?

Example $d = 1$



$$\rho(x, G) \xrightarrow[\varepsilon \to 0]{} \infty$$

a) Smooth functions.

a1. Answer „NO"

a2. Observations $\Pi_\varepsilon \cap \bar{G}$ are not needed. Estimators are local

$$\hat{\lambda}_\varepsilon(x) = a_\varepsilon \sum_{X_i \in \Pi_\varepsilon} K(a_\varepsilon(x - X_i))$$

b) Analytic λ

b1) Unicity theorem says may be „Yes"

b2) The loss of $\Pi_\varepsilon \overline{G}$ may(!) imply the loss in the estimation $\lambda(x)$, $x \in G$.

2. Extrapolation. Π_ε is observed in G. $G \subset R^d$ is a bounded region.
The density $\lambda \in \mathcal{E}(M, \sigma, \rho)$:

$f \in \mathcal{E}(M, \sigma, \rho)$:

$$|f(z_1, \dots z_d)| \leq M \exp\{-\sum \sigma_i |z_i|^{\rho}$$

Set

$$\bar{\rho} = \max \rho_i$$

Result V.1. The extrapolation is possible for z

$$\rho(z, G) \lesssim \left(\ln \frac{1}{z}\right)^{1/\bar{\rho}}$$

and impossible for

$$\rho(z, G) \geq \left(\ln \frac{1}{z}\right)^{1/\bar{\rho}}$$

Theorem V.1. Let $\Lambda = E(M, \sigma, \rho)$. Denote $G_\varepsilon(\alpha)$ the $\left(\ln \frac{1}{\varepsilon}\right)^{\alpha/\bar{\rho}}$ vicinity of G, $0 < \alpha < 1$. There exists $\hat{\lambda}_\varepsilon$ that for all fixed $\rho < \alpha < 1$

$$\sup_{\lambda \in \Lambda} E_\lambda \left\{ \sup_{G_\rho} |\lambda(x) - \hat{\lambda}(x)| \right\} \leq$$

$$\leq C_{\alpha,\rho}\, \varepsilon^{\frac{1-\alpha}{2\alpha}}$$

Theorem V.2. Let $\Lambda = E(M, \sigma, \rho)$ for any $x \in G_\varepsilon(\alpha)$, $\alpha > 1$, and all sufficiently small ε

$$\inf_{\hat{f}} \sup_{\lambda \in \Lambda} E_\lambda |\lambda(x) - \hat{f}|^2 \geq c_0 > 0.$$

c_0 depends on α, M, σ, ρ.

V.3. Let $K \subset \mathbb{R}^d$ be a bounded region. Let

$$\Lambda = \left\{ \lambda : \int_{\mathbb{R}^d} \lambda(x)\, dx = 1, \right.$$

$$\lambda(x) = \frac{1}{(2\pi)^d} \int_K e^{-i(t,x)} \tilde{\lambda}(t)\, dt,$$

$$\left. \tilde{\lambda} \in L_2 \right\}$$

Then $\Lambda \subset E(M, 6, (1, \dots 1))$

If we use all observation Π_ε, $\exists\, \hat{\lambda}_\varepsilon$

$$\sup_\lambda E_\lambda \int_{\mathbb{R}^d} |\lambda(x) - \hat{\lambda}_\varepsilon(x)|^2 dx \asymp \varepsilon$$

Theorem V.3. Let G be a bounded region in $\mathbb{R}^d$. The estimate $\hat{\lambda}_\varepsilon$ is constructed from $\Pi_\varepsilon G$ ($\Pi_\varepsilon \overline{G}$ is lost)

Then

$$\sup_{\lambda \in \Lambda G} E_\lambda \int |\lambda(x) - \hat{\lambda}_\varepsilon(x)|^2 dx \asymp$$

$$\asymp \varepsilon \left(\frac{|\ln \varepsilon|}{\ln|\ln \varepsilon|} \right)^d .$$

Compare

$$\varepsilon \quad \text{and} \quad \varepsilon \cdot \left(\frac{|\ln \varepsilon|}{\ln|\ln \varepsilon|} \right)^d$$