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A Hill type estimator in Non Recurrent Diffusions

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Consider the diffusion process $(X_t, t \geq 0)$ solution of

$$dX(t) = \beta X_t^{\alpha} dt + \sigma dW(t), \quad t \geq 0$$

where

$X_0 = x_0$, $x_0 > 0$
$\beta > 0$ and $\sigma > 0$ are known
$(W(t), t \geq 0)$ is a Wiener process
The parameter of interest is α where

$$-1 < \alpha < 0$$

Our aim .

A continuous time Hill estimator of α in a non recurrent diffusions :
We investigate its consistency and asymptotic normality

Some known results on non-recurrent processes

Parametric estimation on processes

$$dX(t) = \theta a(X_t)dt + \sigma dW(t), \quad t \geq 0$$

where the parameter $\theta \geq 0$

(Remark. If $\theta \leq 0$ and $a(x) = x$ we have X_t positive recurrent).

1. Transient processes

we have

$$\mid X_t \mid \to \infty \quad a.s.$$

at a prescribed rate in 3 cases

1.

$$a(x) = x$$

2.

$$a(x) = cx + r(x)$$

$$\mid r(x) \mid = K(1+ \mid x \mid^{\gamma}), K > 0, \gamma \in (0,1)$$

3.

$$a(x) = \mid x \mid^{\alpha}$$

with $0 \leq \alpha < 1$ known

Consistency and limit law of MLE in transient case :

case 1 and case 2 :

$$e^{\theta c T}(\widehat{\theta}_T - \theta) \Rightarrow \frac{\nu}{\chi}$$

with ν, χ rv's

case 3 :

$$T^{\theta c}(\widehat{\theta}_T - \theta) \Rightarrow N(0, 2\theta)$$

Dietz H. and Kutoyants Y.(2001), Dietz H. (2002), Kutoyants's book (1994), Basawa I.V. and Scott D.J.(1983)

2. Null recurrent processes

Hoepfner-Kutoyants (2003) consider

$$dX(t) = \left(\theta \frac{X_t}{1+X_t^2} + g(X_t)\right) dt + \sigma dW(t), \quad t \geq 0$$

where parameter $\theta \in \Theta = (-\sigma^2/2, \sigma^2/2)$

g a nuisance function

$\sigma > 0$

Results : consistency and limit law of MLE -LAMN condition (local asymptotic mixed normal condition)- efficiency.

$$n^{\frac{\alpha(\theta)}{2}}(\widehat{\theta}_n - \theta) \Rightarrow \quad mixture \quad of \quad normals$$

where

$$\alpha(\theta) = \frac{1}{2} - \frac{\theta}{\sigma^2}$$

Our problem :

Estimation of the parameter α observing a trajectory $(X_t, t \in [0,T])$ of

$$dX(t) = \beta X_t^{\alpha} dt + \sigma dW(t), \quad t \geq 0, \quad x_0 > 0$$

where $\beta > 0$ and $\sigma > 0$ are known

The parameter α :

$$-1 < \alpha < 0$$

A Hill estimator $\widehat{\alpha}_T$:

we state consistency and a limit law

Some review on the Hill estimator

In extreme value theory.

1. IID case : let F_θ be a parametric distribution

$Z_1, ..., Z_n$ is a sample of F_θ

the order statistics

$$Z_{(1)}, ..., Z_{(n)}$$

Taking the k largest order statistics

$$Z_{(n-k+1)}, ..., Z_{(n)}$$

A Hill estimator is defined by

$$\widehat{\theta}_n = \frac{1}{k-1} \sum_{i=1}^{k-1} \log(Z_{(n-i+1)}) - \log(Z_{(n-k+1)})$$

$k = k(n) \to \infty$, $k(n)/n \to 0$

results : LAN condition-asymptotic properties-efficiency

(Hill (1975), Hall (1982) Marohn (1995), Falk (1995), Wei (1995) etc ...)

2. **Stable processes**

for stable processes

$$X_t = \int_0^t \int_0^\infty x\mu(ds, dx)$$

where

μ is a Poisson random measure

intensity $\mu_{\alpha,\xi}$ on $(0, \infty) \times (0, \infty)$

$$\mu_{\alpha,\xi}(ds, dx) = \xi\alpha x^{-\alpha-1} ds dx$$

The parameters are $\alpha > 0$ and $\xi > 0$

The Hill estimator is considered in Hoepfner and Jacod (1994), Hoepfner (1994), Hoepfner (1998)) :

results: LAN condition-consistency- a limit law - efficiency

The Hill estimator $\widehat{\alpha}_T$
Notations
we observe $(X_t, t \in [0, T])$ from

$$dX(t) = \beta X_t^{\alpha} dt + \sigma dW(t), \quad t \geq 0, \quad x_0 > 0$$

where $\beta > 0, \ -1 < \alpha < 0$ and $\sigma > 0$
For a decreasing numbers (λ_i) :

$$0 < \lambda_{i+1} < \lambda_i < 1$$

define

$$X_{\lambda_i}^{(T)} := X_{\lambda_i T}$$

for $i = 1, .., k$ and k is given.

Consider the "k largest" observations of the process (X_t) :

$$X_{\lambda_k}^{(T)}, \quad X_{\lambda_{k-1}}^{(T)}, \quad .. \quad , \quad X_{\lambda_1}^{(T)}$$

Set

$$\gamma := \frac{1}{1-\alpha}$$

Define a Hill estimator of γ by

$$\widehat{\gamma}_n = \frac{1}{k} \sum_{i=1}^{k} \frac{\log X_{\lambda_1}^{(T)} - \log X_{\lambda_i}^{(T)})}{\log \lambda_1 - \log \lambda_i}$$

or equivalently

$$\widehat{\gamma}_n = \frac{1}{k} \sum_{i=1}^{k} \frac{\log(\frac{X_{\lambda_1}^{(T)}}{X_{\lambda_i}^{(T)}})}{\log(\frac{\lambda_1}{\lambda_i})}$$

The Hill estimator of α is defined by

$$\hat{\alpha}_T := \frac{\widehat{\gamma}_T - 1}{\widehat{\gamma}_T}$$

Remark
in our setting k doesn't depend on T contrarily to case. We can explain this by the behavior of the trajectories of the process (which go to infinity a.s.) so we can said that we observe "often " extreme values of the process.

Results

Asymptotic behavior of X_t
Th. 5. 17 in Gikhman-Skorohod (1977)
The diffusion process $(X_t, t \geq 0)$ solution of

$$dX(t) = \beta X_t^{\alpha} dt + \sigma dW(t), \quad t \geq 0, \quad x_0 > 0$$

with $\beta > 0$, $-1 < \alpha < 0$ and $\sigma > 0$
satisfies

$$X_t \longrightarrow \infty, \quad a.s. \quad t \longrightarrow \infty$$

Precisely there exists $C_{\alpha,\beta} > 0$ such that

$$X_t \asymp C_{\alpha,\beta} t^{\frac{1}{2-\alpha}}, \quad a.s. \quad t \to \infty$$

Theorem .
The Hill estimator $\widehat{\gamma}_T$ *satisfies as* $T \to \infty$

$$\widehat{\gamma}_T \to \gamma \quad in \quad probability$$

and

$$T^{\frac{1}{2}(2\gamma-1)}(\widehat{\gamma}_T - \gamma) \Rightarrow N(0, (\frac{\gamma}{\beta})^{2\gamma} \sum_{i=1}^{k-1} i^2 \frac{\lambda_i - \lambda_{i+1}}{\lambda_{i+1}^{2\gamma}})$$

Remark :
we obtain the rate: for $-1 < \alpha < 0$

$$\frac{1}{2}(2\gamma - 1) < \frac{1}{2}$$

Corollary.

The Hill estimator $\widehat{\alpha}_T$ satisfies as $T \to \infty$

$$\widehat{\alpha}_T \to \alpha \quad in \quad probability$$

and

$$T^{\frac{1}{2}\frac{1+\alpha}{1-\alpha}}(\widehat{\alpha}_T - \alpha) \Rightarrow N(0, \frac{1}{\gamma^4}(\frac{\gamma}{\beta})^{2\gamma} \sum_{i=1}^{k-1} i^2 \frac{\lambda_i - \lambda_{i+1}}{\lambda_{i+1}^{2\gamma}})$$

Remark .

1.Asymptotic behavior of X_t : plugging the Hill estimator in asymptotic behavior we obtain

$$X_t \asymp t^{\frac{1}{2-\hat{\alpha}_T}}, \quad a.s. \quad t \to \infty$$

which can be used to construct a "trajectory fitting estimator" for a parameter θ in the drift θX_t^{α} (Dietz H. (2002)for $0 < \alpha < 1$)

2. For k fixed how chose λ_i such that the quantity $\sum_{i=1}^{k-1} i^2 \frac{\lambda_i - \lambda_{i+1}}{\lambda_{i+1}^{2\gamma}}$ become small ?

3. A choice of (λ_i, k) giving a small asymptotic variance ?

4. MLE for the parameters (θ, α) ?

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