Adaptive Bayes type estimation of ergodic diffusion processes based on sampled data

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Plan of today's talk

1. Introduction: Bayesian estimation in the case when $nh_n^2 \to 0$

2. Two kinds of adaptive Bayes type estimators with convergence of moments in the case when $nh_n^p \to 0$ $(p \geq 2)$

3. Example and simulation results

1. Introduction

We consider a d-dimensional ergodic diffusion process defined by the following stochastic differential equation

$$dX_t = a(X_t, \theta_2)dt + b(X_t, \theta_1)dw_t, \quad X_0 = x_0 \tag{1}$$

where w is an r-dimensional standard Wiener process,

x_0 is a deterministic initial condition,

$\theta = (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 = \Theta$ with Θ_1 and Θ_2 being compact convex subsets of $\mathbf{R}^{p_1}$ and $\mathbf{R}^{p_2}$, respectively.

Moreover, $a : \mathbf{R}^d \times \Theta_2 \to \mathbf{R}^d$ and $b : \mathbf{R}^d \times \Theta_1 \to \mathbf{R}^d \otimes \mathbf{R}^r$.

$\theta^* = (\theta_1^*, \theta_2^*)$ is the true value of θ and we assume that $\theta^* \in \mathrm{Int}(\Theta)$.

The data are discrete observations $\mathbf{X}_n = (X_{t_i^n})_{0 \le i \le n}$ with $t_i^n = ih_n$ and $t_n^n = nh_n = T_n$.

Let p be an integer value and $p \geq 2$.

We will consider the situation when $h_n \to 0$ and $nh_n^p \to 0$ as $n \to \infty$, and there exists $\epsilon_0 \in (0, (p-1)/p)$ such that $n^{\epsilon_0} \leq nh_n$.

The Bayesian inference for continuously observed diffusion processes has been investigated by many researchers, e.g.,

Basawa and Prakasa Rao (1980),

Kutoyants (1984, 1994, 2004) studied Ibragimov-Has'minskii's scheme for stochastic processes

Yoshida (1993, 2005).

On the other hand, as for the Bayes type estimation of discretely observed diffusion processes,

Yoshida (2005) considered general results of the simultaneous Bayes type estimation and the adaptive Bayes type estimation.
As an example, the Bayes type estimators for discretely observed ergodic diffusion processes in the case when $nh_n^2 \to 0$.

Ogihara and Yoshida (2010) ergodic jump-diffusions.

Uchida and Yoshida (2010) presented the Bayes type estimator of the volatility parameter based on high frequency data on the fixed time interval.

Adaptive Bayes type estimator in the case when $nh_n^2 \to 0$ (Yoshida (2005)).

$$dX_t = a(X_t, \theta_2)dt + b(X_t, \theta_1)dw_t, \quad X_0 = x_0$$

The prior densities $\pi_1(\theta_1)$ and $\pi_2(\theta_2)$ are assumed to be continuous and to satisfy that $0 < \inf_{\theta_i \in \Theta_i} \pi_i(\theta_i) \leq \sup_{\theta_i \in \Theta_i} \pi_i(\theta_i) < \infty$ for $i = 1, 2$.

Set $B(x, \theta_1) = bb^\star(x, \theta_1)$, where $\star$ denotes the transpose. Let $\Delta X_i = X_{t_i^n} - X_{t_{i-1}^n}$, $B_{i-1}(\theta_1) = B(X_{t_{i-1}^n}, \theta_1)$ and $a_{i-1}(\theta_2) = a(X_{t_{i-1}^n}, \theta_2)$.

The utility functions are as follows:

$$U_n^{(0)}(\theta_1) = -\frac{1}{2}\sum_{i=1}^n \left\{ h_n^{-1} B_{i-1}^{-1}(\theta_1)[(\Delta X_i)^{\otimes 2}] + \log\det(B_{i-1}(\theta_1)) \right\},$$

$$U_n^{(1)}(\theta_1, \theta_2) = -\frac{1}{2}\sum_{i=1}^n \left\{ h_n^{-1} B_{i-1}^{-1}(\theta_1)[(\Delta X_i - h_n a_{i-1}(\theta_2))^{\otimes 2}] + \log\det(B_{i-1}(\theta_1)) \right\}.$$

The adaptive Bayes type estimators $\tilde{\theta}_{1,n}$ and $\tilde{\theta}_{2,n}$ are defined as

$$\tilde{\theta}_{1,n} = \frac{\int_{\Theta_1} \theta_1 \exp\left\{U_n^{(0)}(\theta_1)\right\}\pi_1(\theta_1)d\theta_1}{\int_{\Theta_1} \exp\left\{U_n^{(0)}(\theta_1)\right\}\pi_1(\theta_1)d\theta_1},$$

$$\tilde{\theta}_{2,n} = \frac{\int_{\Theta_2} \theta_2 \exp\left\{U_n^{(1)}(\tilde{\theta}_{1,n},\theta_2)\right\}\pi_2(\theta_2)d\theta_2}{\int_{\Theta_2} \exp\left\{U_n^{(1)}(\tilde{\theta}_{1,n},\theta_2)\right\}\pi_2(\theta_2)d\theta_2}.$$

Under some regularity conditions, as $nh_n^2 \to 0$,

$$(\sqrt{n}(\tilde{\theta}_{1,n}-\theta_1^*), \sqrt{nh_n}(\tilde{\theta}_{2,n}-\theta_2^*)) \xrightarrow{d} (\zeta_1,\zeta_2) \sim N_{p_1+p_2}(0, \mathrm{diag}[\Gamma_1(\theta_1^*)^{-1}, \Gamma_2(\theta^*)^{-1}]),$$

and

$$E_{\theta^*}[f(\sqrt{n}(\widehat{\theta}_{1,p,n}-\theta_1^*), \sqrt{nh_n}(\widehat{\theta}_{2,p,n}-\theta_2^*))] \to \mathbb{E}[f(\zeta_1,\zeta_2)]$$

for all continuous functions f of at most polynomial growth, where

$$\Gamma_1(\theta_1^*)[u_1,u_1] = \frac{1}{2}\int_{\mathbf{R}^d} tr\{B^{-1}(\partial_{\theta_1}B)B^{-1}(\partial_{\theta_1}B)(x,\theta_1^*)[u_1^{\otimes 2}]\mu_{\theta^*}(dx)$$

$$\Gamma_2(\theta^*)[u_2,u_2] = \int_{\mathbf{R}^d} B(x,\theta_1^*)^{-1}[\partial_{\theta_2}a(x,\theta_2^*)[u_2], \partial_{\theta_2}a(x,\theta_2^*)[u_2]]\mu_{\theta^*}(dx)$$

for $u_1 \in \mathbf{R}^{p_1}$ and $u_2 \in \mathbf{R}^{p_2}$.

Polynomial type large deviation inequality (key lemma)

Statistical random fields are as follows:

$$\mathbb{Z}_n^{(0)}(u_1;\theta_1^*) = \exp\left\{U_n^{(0)}(\theta_1^* + \frac{u_1}{\sqrt{n}}) - U_n^{(0)}(\theta_1^*)\right\},$$

$$\tilde{\mathbb{Z}}_n^{(1)}(u_2;\theta_2^*) = \exp\left\{U_n^{(1)}(\tilde{\theta}_{1,n}, \theta_2^* + \frac{u_2}{\sqrt{nh_n}}) - U_n^{(1)}(\tilde{\theta}_{1,n}, \theta_2^*)\right\},$$

where

$$U_n^{(0)}(\theta_1) = -\frac{1}{2}\sum_{i=1}^{n}\left\{h_n^{-1}B_{i-1}^{-1}(\theta_1)[(\Delta X_i)^{\otimes 2}] + \log\det(B_{i-1}(\theta_1))\right\},$$

$$U_n^{(1)}(\theta_1,\theta_2) = -\frac{1}{2}\sum_{i=1}^{n}\left\{h_n^{-1}B_{i-1}^{-1}(\theta_1)[(\Delta X_i - h_n a_{i-1}(\theta_2))^{\otimes 2}] + \log\det(B_{i-1}(\theta_1))\right\}.$$

Under some regularity conditions, for any $L > 0$, there exists $C_L > 0$ such that

$$P_{\theta^*}\left[\sup_{u_1 \in V_n(r)} \mathbb{Z}_n^{(0)}(u_1; \theta_1^*) \geq e^{-r}\right] \leq \frac{C_L}{r^L},$$
$$P_{\theta^*}\left[\sup_{u_2 \in \tilde{V}_n(r)} \tilde{\mathbb{Z}}_n^{(1)}(u_2; \theta_2^*) \geq e^{-r}\right] \leq \frac{C_L}{r^L}$$

for all $n \in \mathbf{N}$ and $r > 0$, where

$$\mathbb{U}_n = \left\{u_1 \in \mathbf{R}^{p_1} \mid \theta_1^* + \frac{u_1}{\sqrt{n}} \in \Theta_1\right\}, \quad V_n(r) = \{u_1 \in \mathbb{U}_n \mid r \leq |u_1|\},$$
$$\tilde{\mathbb{U}}_n = \left\{u_2 \in \mathbf{R}^{p_2} \mid \theta_2^* + \frac{u_2}{\sqrt{nh_n}} \in \Theta_2\right\}, \quad \tilde{V}_n(r) = \{u_2 \in \tilde{\mathbb{U}}_n \mid r \leq |u_2|\}$$

In this talk, our goal is

to obtain two kinds of adaptive Bayes type estimators in the case when $nh_n^p \to 0$ $(p \geq 2)$ and

to show the asymptotic properties (asymptotic normality and convergence of moments) of the adaptive Bayes type estimators.

Utility function (Kessler's contrast function (1997)) of d-dim. diffusions for $nh_n^p \to 0$

Let $p \geq 2$, $k_0 = \left[\frac{p}{2}\right]$. Let $B_{i-1}(\theta_1) = B(X_{t_{i-1}^n}, \theta)$.

In the same way as in Kessler (1997), the utility function $U_{p,n}(\theta)$ is defined as

$$U_{p,n}(\theta) = -\frac{1}{2}\sum_{i=1}^{n}\left\{h_n^{-1}\left\{\sum_{j=0}^{k_0} h_n^j D_{i-1}^{(j)}(\theta)\right\}\left[(X_{t_i^n} - r_{i-1}^{(k_0)}(h_n,\theta))^{\otimes 2}\right] + \sum_{j=0}^{k_0} h_n^j E_{i-1}^{(j)}(\theta)\right\}.$$

where $(r_{i-1}^{(k_0)}(h_n,\theta))_m = \sum_{j=0}^{k_0} \frac{h_n^j}{j!} L_\theta^j f_k(X_{t_{i-1}^n})$, $f_m(x) = x_m$ and for example,

$$\begin{aligned}
D_{i-1}^{(0)}(\theta) &= B_{i-1}^{-1}(\theta_1), \quad E_{i-1}^{(0)}(\theta) = \log\det(B_{i-1}(\theta_1)),\\
D_{i-1}^{(1)}(\theta) &= -B_{i-1}^{-1}(\theta_1)\gamma^{(2)}(X_{t_{i-1}^n},\theta), \quad E_{i-1}^{(1)}(\theta) = tr[B_{i-1}^{-1}(\theta_1)\gamma^{(2)}(X_{t_{i-1}^n},\theta)],\\
\gamma_{kl}^{(2)}(x,\theta) &= \frac{1}{2}\left\{L_\theta B_{kl}(x,\theta_1) + \sum_{j=1}^{d}\{(\partial_{x_j} a_k(x,\theta_2))B_{jl}(x,\theta_1) + (\partial_{x_j} a_l(x,\theta_2))B_{jk}(x,\theta_1)\}\right\}
\end{aligned}$$

for $k, l = 1, \ldots, d$.

2. Assumptions and notation

Let $C_{\uparrow}^{k,l}(\mathbf{R}^d \times \Theta; \mathbf{R}^d)$ denote the space of all functions f satisfying the following conditions: (i) $f(x,\theta)$ is an $\mathbf{R}^d$-valued function on $\mathbf{R}^d \times \Theta$, (ii) $f(x,\theta)$ is continuously differentiable with respect to x up to order k for all θ, and their derivatives up to order k are of polynomial growth in x uniformly in θ. (iii) for $|\mathbf{n}| = 0, 1, \ldots, k$, $\partial^{\mathbf{n}} f(x,\theta)$ is continuously differentiable with respect to θ up to order l for all x. Moreover, for $|\nu| = 1, \ldots, l$ and $|\mathbf{n}| = 0, 1, \ldots, k$, $\delta^{\nu}\partial^{\mathbf{n}} f(x,\theta)$ is of polynomial growth in x uniformly in θ.

Let $\mathcal{F}_{\uparrow}(\mathbf{R}^d)$ be the space of all measurable functions f satisfying that $f(x)$ is an $\mathbf{R}$-valued function on $\mathbf{R}^d$ with polynomial growth in x.

Let $\stackrel{p}{\to}$ and $\stackrel{d}{\to}$ be the convergence in probability and the convergence in distribution, respectively.

Let L_θ be the infinitesimal generator of the diffusion (1): $L_\theta = \sum_{i=1}^d a_i(x,\theta_2)\partial_i + \frac{1}{2}\sum_{i,j=1}^d B_{ij}(x,\theta_1)\partial_i\partial_j$.

We make the following assumptions.

$[A1]$ (i) There exists $K > 0$ such that for all $x, y \in \mathbf{R}^d$,

$$\sup_{\theta_2 \in \Theta_2} |a(x,\theta_2) - a(y,\theta_2)| + \sup_{\theta_1 \in \Theta_1} |b(x,\theta_1) - b(y,\theta_1)| \leq K|x-y|.$$

(ii) $\inf_{x,\theta_1} \det(B(x,\theta_1)) > 0$.
(iii) There exists a unique invariant probability measure μ_{θ^*} of X_t and for any $f \in \mathcal{F}_\uparrow(\mathbf{R}^d)$ satisfying $\int_{\mathbf{R}^d} |f(x)|\mu_{\theta^*}(dx) < \infty$, as $T \to \infty$,

$$\frac{1}{T}\int_0^T f(X_t)dt \xrightarrow{p} \int_{\mathbf{R}^d} f(x)\mu_{\theta^*}(dx),$$

(iv) $\sup_t E[|X_t|^M] < \infty$ for all $M > 0$.
(v) For any $g \in \mathcal{F}_\uparrow(\mathbf{R}^d)$ satisfying $\int_{\mathbf{R}^d} g(x)\mu_{\theta^*}(dx) = 0$, there exist $G(x)$, $\partial_{x_i} G(x) \in \mathcal{F}_\uparrow(\mathbf{R}^d)$ $(i = 1, \ldots, d)$ such that for all x,

$$L_{\theta^*} G(x) = -g(x).$$

$[A2](k,l)$ $a \in C_\uparrow^{k,4}(\mathbf{R}^d \times \Theta_2; \mathbf{R}^d)$. $b \in C_\uparrow^{l,4}(\mathbf{R}^d \times \Theta_1; \mathbf{R}^d \otimes \mathbf{R}^r)$.

Remark 1. *(i) For a sufficient condition for* $[A1]$-(v)*, see Pardoux and Veretennikov (2001). For example, in addition to* $[A1]$-(i)-(ii)*, we assume that* $\sup_{x,\theta_1}|B(x,\theta_1)| < \infty$ *and that there exist* $c_0 > 0$*,* $M_0 > 0$ *and* $\alpha \geq 0$ *such that for all* θ_2*,*

$$\frac{x^* a(x,\theta_2)}{|x|} \leq -c_0|x|^\alpha \qquad \text{for all } x \text{ satisfying } |x| \geq M_0.$$

Then, $[A1]$-(v) *holds with* $[A1]$-(iii)-(iv)*.*
(ii) Let $\epsilon_1 = \epsilon_0/(2(p-1))$ *and* $f \in C_{\uparrow}^{1,1}(\mathbf{R}^d \times \Theta)$*. Under* $[A1]$*,*

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\sup_{\theta\in\Theta}\left(n^{\epsilon_1}\left|\frac{1}{n}\sum_{i=1}^n f(X_{t_{i-1}^n},\theta) - \int_{\mathbf{R}^d} f(x,\theta)\mu_{\theta^*}(dx)\right|\right)^M\right] < \infty$$

for all $M > 0$*, see Uchida (2010).*

3. Adaptive Bayes type estimation

First of all, we consider the initial Bayes type estimator of θ_1.

The prior density $\pi_1(\theta_1)$ is assumed to be continuous and to satisfy that

$$0 < \inf_{\theta_1 \in \Theta_1} \pi_1(\theta_1) \leq \sup_{\theta_1 \in \Theta_1} \pi_1(\theta_1) < \infty.$$

Let

$$\begin{aligned} U_n^{(0)}(\theta_1) &= -\frac{1}{2}\sum_{i=1}^{n}\left\{h_n^{-1}B_{i-1}^{-1}(\theta_1)[(\Delta X_i)^{\otimes 2}] + \log\det(B_{i-1}(\theta_1))\right\}, \\ \mathbb{H}_{p,n}^{(0)}(\theta_1) &= \frac{1}{n^{1-\frac{2}{p}}}U_n^{(0)}(\theta_1). \end{aligned}$$

The initial Bayes type estimator $\tilde{\theta}_{1,p,n}^{(0)}$ is defined as

$$\begin{aligned} \tilde{\theta}_{1,p,n}^{(0)} &= \frac{\int_{\Theta_1}\theta_1 \exp\left\{\mathbb{H}_{p,n}^{(0)}(\theta_1)\right\}\pi_1(\theta_1)d\theta_1}{\int_{\Theta_1}\exp\left\{\mathbb{H}_{p,n}^{(0)}(\theta_1)\right\}\pi_1(\theta_1)d\theta_1} \\ &= \frac{\int_{\Theta_1}\theta_1 \exp\left\{\frac{1}{n^{1-\frac{2}{p}}}U_n^{(0)}(\theta_1)\right\}\pi_1(\theta_1)d\theta_1}{\int_{\Theta_1}\exp\left\{\frac{1}{n^{1-\frac{2}{p}}}U_n^{(0)}(\theta_1)\right\}\pi_1(\theta_1)d\theta_1}. \end{aligned}$$

Let

$$\mathbb{Y}_n(\theta_1) = \frac{1}{n^{\frac{2}{p}}}\left(\mathbb{H}^{(0)}_{p,n}(\theta_1) - \mathbb{H}^{(0)}_{p,n}(\theta_1^*)\right) = \frac{1}{n}\left(U^{(0)}_n(\theta_1) - U^{(0)}_n(\theta_1^*)\right),$$

$$\mathbb{Y}(\theta_1) = -\frac{1}{2}\int_{\mathbf{R}^d}\left\{tr\left[B(x,\theta_1)^{-1}B(x,\theta_1^*) - I_d\right] + \log\frac{\det(B(x,\theta_1))}{\det(B(x,\theta_1^*))}\right\}\mu_{\theta^*}(dx).$$

We make the assumption as follows.

$[A3]$ There exists a positive constant χ such that $\mathbb{Y}(\theta_1) \le -\chi|\theta_1 - \theta_1^*|^2$ for all $\theta_1 \in \Theta_1$.

Proposition 1. *Let* $p \ge 2$. *Assume* $[A1]$, $[A2](2,2)$ *and* $[A3]$. *Then, for all* $M > 0$, *as* $nh_n^p \to 0$,

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|n^{\frac{1}{p}}(\tilde{\theta}^{(0)}_{1,p,n} - \theta_1^*)\right|^M\right] < \infty.$$

Next, we consider the adaptive Bayes type estimators.

The prior density $\pi_2(\theta_2)$ is assumed to be continuous and to satisfy that $0 < \inf_{\theta_2 \in \Theta_2} \pi_2(\theta_2) \leq \sup_{\theta_2 \in \Theta_2} \pi_2(\theta_2) < \infty$.

In case that $p = 2$, the adaptive Bayes type estimators $\tilde{\theta}_{1,p,n}^{(0)}$ and $\tilde{\theta}_{2,p,n}^{(1)}$ are defined as

$$\begin{aligned}
\tilde{\theta}_{1,p,n}^{(0)} &= \frac{\int_{\Theta_1} \theta_1 \exp\left\{U_n^{(0)}(\theta_1)\right\} \pi_1(\theta_1) d\theta_1}{\int_{\Theta_1} \exp\left\{U_n^{(0)}(\theta_1)\right\} \pi_1(\theta_1) d\theta_1}, \\
\tilde{\theta}_{2,p,n}^{(1)} &= \frac{\int_{\Theta_2} \theta_2 \exp\left\{U_n^{(1)}(\tilde{\theta}_{1,p,n}^{(0)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}{\int_{\Theta_2} \exp\left\{U_n^{(1)}(\tilde{\theta}_{1,p,n}^{(0)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2},
\end{aligned}$$

where

$$U_n^{(1)}(\theta_1, \theta_2) = -\frac{1}{2} \sum_{i=1}^{n} \left\{ h_n^{-1} B_{i-1}^{-1}(\theta_1)[(\Delta X_i - h_n a_{i-1}(\theta_2))^{\otimes 2}] + \log\det(B_{i-1}(\theta_1)) \right\}.$$

Let $p \geq 3$ and $k_0 = \left[\frac{p}{2}\right]$.

In the same way as in Kessler (1997), the utility function $U_{p,n}(\theta)$ is defined as

$$U_{p,n}(\theta) \quad = \quad -\frac{1}{2}\sum_{i=1}^{n}\left\{h_n^{-1}\left\{\sum_{j=0}^{k_0} h_n^j D_{i-1}^{(j)}(\theta)\right\}\left[(X_{t_i^n} - r_{i-1}^{(k_0)}(h_n,\theta))^{\otimes 2}\right] + \sum_{j=0}^{k_0} h_n^j E_{i-1}^{(j)}(\theta)\right\}.$$

Let $p \geq 3$, $k_0 = \left[\frac{p}{2}\right]$ and $l_0 = \left[\frac{p-1}{2}\right]$. Note that $l_0 \leq k_0 \leq l_0 + 1$.

In case that $p \geq 3$, the adaptive Bayes type estimators $\tilde{\theta}_{1,p,n}^{(l_0)}$ and $\tilde{\theta}_{2,p,n}^{(k_0)}$ are defined as follows: for $k = 1, 2, \ldots, l_0$,

$$\begin{aligned}
\tilde{\theta}_{1,p,n}^{(k-1)} &= \frac{\int_{\Theta_1} \theta_1 \exp\left\{\mathbb{H}_{p,n}^{(k-1)}(\theta_1, \tilde{\theta}_{2,p,n}^{(k-1)})\right\} \pi_1(\theta_1) d\theta_1}{\int_{\Theta_1} \exp\left\{\mathbb{H}_{p,n}^{(k-1)}(\theta_1, \tilde{\theta}_{2,p,n}^{(k-1)})\right\} \pi_1(\theta_1) d\theta_1}, \\
\tilde{\theta}_{2,p,n}^{(k)} &= \frac{\int_{\Theta_2} \theta_2 \exp\left\{\tilde{\mathbb{H}}_{p,n}^{(k)}(\tilde{\theta}_{1,p,n}^{(k-1)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}{\int_{\Theta_2} \exp\left\{\tilde{\mathbb{H}}_{p,n}^{(k)}(\tilde{\theta}_{1,p,n}^{(k-1)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}, \\
\tilde{\theta}_{1,p,n}^{(l_0)} &= \frac{\int_{\Theta_1} \theta_1 \exp\left\{U_{p,n}(\theta_1, \tilde{\theta}_{2,p,n}^{(l_0)})\right\} \pi_1(\theta_1) d\theta_1}{\int_{\Theta_1} \exp\left\{U_{p,n}(\theta_1, \tilde{\theta}_{2,p,n}^{(l_0)})\right\} \pi_1(\theta_1) d\theta_1}, \\
\tilde{\theta}_{2,p,n}^{(l_0+1)} &= \frac{\int_{\Theta_2} \theta_2 \exp\left\{U_{p,n}(\tilde{\theta}_{1,p,n}^{(l_0)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}{\int_{\Theta_2} \exp\left\{U_{p,n}(\tilde{\theta}_{1,p,n}^{(l_0)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2},
\end{aligned}$$

where for $k = 1, 2, \ldots, l_0$,

$$\begin{aligned}
\mathbb{H}_{p,n}^{(0)}(\theta_1, \theta_2) &= \frac{1}{n^{1-\frac{2}{p}}} U_n^{(0)}(\theta_1), \\
\mathbb{H}_{p,n}^{(k)}(\theta_1, \theta_2) &= \frac{1}{n^{1-\frac{2(k+1)}{p}}} U_{p,n}(\theta_1, \theta_2), \\
\tilde{\mathbb{H}}_{p,n}^{(k)}(\theta_1, \theta_2) &= \frac{1}{(nh_n)^{1-\frac{2k}{p-1}}} U_{p,n}(\theta_1, \theta_2).
\end{aligned}$$

For example, we consider the case when $p = 4$ $(nh_n^4 \to 0)$. Noting that $l_0 = 1$ and $k_0 = l_0 + 1 = 2$, one has the following steps.
Step 0: Obtain the initial estimator $\widetilde{\theta}^{(0)}_{1,p,n}$ of θ_1:

$$\widetilde{\theta}^{(0)}_{1,p,n} = \frac{\int_{\Theta_1} \theta_1 \exp\left\{\frac{1}{n^{1-\frac{2}{p}}} U_n^{(0)}(\theta_1)\right\} \pi_1(\theta_1) d\theta_1}{\int_{\Theta_1} \exp\left\{\frac{1}{n^{1-\frac{2}{p}}} U_n^{(0)}(\theta_1))\right\} \pi_1(\theta_1) d\theta_1}.$$

Step 1: Obtain the 1st-step adaptive estimator $\widetilde{\theta}^{(1)}_{2,p,n}$ of θ_2:

$$\widetilde{\theta}^{(1)}_{2,p,n} = \frac{\int_{\Theta_2} \theta_2 \exp\left\{\frac{1}{(nh_n)^{1-\frac{2}{p-1}}} U_{p,n}(\widetilde{\theta}^{(0)}_{1,p,n}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}{\int_{\Theta_2} \exp\left\{\frac{1}{(nh_n)^{1-\frac{2}{p-1}}} U_{p,n}(\widetilde{\theta}^{(0)}_{1,p,n}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}.$$

Next, we obtain the 1st-step adaptive estimator $\widetilde{\theta}^{(1)}_{1,p,n}$ of θ_1:

$$\widetilde{\theta}^{(1)}_{1,p,n} = \frac{\int_{\Theta_1} \theta_1 \exp\left\{U_{p,n}(\theta_1, \widetilde{\theta}^{(1)}_{2,p,n})\right\} \pi_1(\theta_1) d\theta_1}{\int_{\Theta_1} \exp\left\{U_{p,n}(\theta_1, \widetilde{\theta}^{(1)}_{2,p,n})\right\} \pi_1(\theta_1) d\theta_1}.$$

Here we get $\widehat{\theta}^{(l_0)}_{1,p,n}$ with $l_0 = 1$.
Step 2: Obtain the 2nd-step adaptive estimator $\widehat{\theta}^{(2)}_{2,p,n}$ of θ_2:

$$\widetilde{\theta}^{(2)}_{2,p,n} = \frac{\int_{\Theta_2} \theta_2 \exp\left\{U_{p,n}(\widetilde{\theta}^{(1)}_{1,p,n}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}{\int_{\Theta_2} \exp\left\{U_{p,n}(\widetilde{\theta}^{(1)}_{1,p,n}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}.$$

Here we get $\widehat{\theta}^{(k_0)}_{2,p,n}$ with $k_0 = 2$.

Let

$$\tilde{\mathbb{Y}}(\theta_2) = -\frac{1}{2}\int_{\mathbf{R}^d} B(x,\theta_1^*)^{-1}[(a(x,\theta_2) - a(x,\theta_2^*))^{\otimes 2}]\mu_{\theta^*}(dx).$$

We make the following assumption.

$[A4]$ There exists a positive constant $\tilde{\chi}$ such that $\tilde{\mathbb{Y}}(\theta_2) \leq -\tilde{\chi}|\theta_2 - \theta_2^*|^2$ for all $\theta_2 \in \Theta_2$.

Proposition 2. *Let $k \in \mathbf{N}$, $p \geq 2k+1$ and $k_0 = \left[\frac{p}{2}\right]$. Assume* $[A1]$, $[A2](2k_0, 2k_0+1)$, $[A3]$ *and* $[A4]$. *Then, for all $M > 0$, as $nh_n^p \to 0$,*

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|(nh_n)^{\frac{k}{p-1}}(\tilde{\theta}^{(k)}_{2,p,n} - \theta^*_2)\right|^M\right] < \infty.$$

Remark 2. *(i) For example, let $p = 5$. Since $p \geq 3$ ($k = 1$), Proposition 2 implies that for all $M > 0$,*

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|(nh_n)^{\frac{1}{4}}(\tilde{\theta}^{(1)}_{2,p,n} - \theta^*_2)\right|^M\right] < \infty.$$

Furthermore, since $p \geq 5$ ($k = 2$), Proposition 2 yields that for all $M > 0$,

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|(nh_n)^{\frac{1}{2}}(\tilde{\theta}^{(2)}_{2,p,n} - \theta^*_2)\right|^M\right] < \infty.$$

(ii) By Proposition 2, $\tilde{\theta}^{(k+1)}_{2,p,n}$ improves $\tilde{\theta}^{(k)}_{2,p,n}$ in the sense that the rate of convergence for $\tilde{\theta}^{(k+1)}_{2,p,n}$ is the one for $\tilde{\theta}^{(k)}_{2,p,n}$ plus $(nh_n)^{\frac{1}{p-1}}$. Moreover, when $p = 2k+1$, $\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|\sqrt{nh_n}(\tilde{\theta}^{(k)}_{2,p,n} - \theta^*_2)\right|^M\right] < \infty$ for all $M > 0$. It is easy to show that for $p = 2k+1$, $\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|\sqrt{n}(\tilde{\theta}^{(k)}_{1,p,n} - \theta^*_1)\right|^M\right] < \infty$ for all $M > 0$.

Proposition 3. *Let $k \in \mathbf{N}$, $p \geq 2(k+1)$ and $k_0 = \left[\frac{p}{2}\right]$. Assume $[A1]$, $[A2](2k_0, 2k_0+1)$, $[A3]$ and $[A4]$. Then, for all $M > 0$, as $nh_n^p \to 0$,*

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|n^{\frac{k+1}{p}}(\tilde{\theta}_{1,p,n}^{(k)} - \theta_1^*)\right|^M\right] < \infty.$$

Remark 3. *(i) For example, let $p = 6$. Since $p \geq 4$ $(k = 1)$, Proposition 3 yields that for all $M > 0$,*

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|n^{\frac{1}{3}}(\tilde{\theta}_{1,p,n}^{(1)} - \theta_1^*)\right|^M\right] < \infty.$$

Furthermore, since $p \geq 6$ $(k = 2)$, Proposition 3 implies that for all $M > 0$,

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|n^{\frac{1}{2}}(\tilde{\theta}_{1,p,n}^{(2)} - \theta_1^*)\right|^M\right] < \infty.$$

(ii) By Proposition 3, we see that $\tilde{\theta}_{1,p,n}^{(k+1)}$ improves $\tilde{\theta}_{1,p,n}^{(k)}$ in the sense that the rate of convergence for $\tilde{\theta}_{1,p,n}^{(k+1)}$ is the one for $\tilde{\theta}_{1,p,n}^{(k)}$ plus $n^{\frac{1}{p}}$. Moreover, Proposition 3 together with Proposition 1 yields that for $p = 2k$ $(k \in \mathbf{N})$, $\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|\sqrt{n}(\tilde{\theta}_{1,p,n}^{(k-1)} - \theta_1^*)\right|^M\right] < \infty$ for all $M > 0$. It is easy to show that for $p = 2k$, $\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|\sqrt{nh_n}(\tilde{\theta}_{2,p,n}^{(k)} - \theta_2^*)\right|^M\right] < \infty$ for all $M > 0$.

Proposition 4. *Let* $p \geq 2$, $l_0 = \left[\frac{p-1}{2}\right]$ *and* $k_0 = \left[\frac{p}{2}\right]$. *Assume* $[A1]$, $[A2](2k_0, 2k_0+1)$, $[A3]$ *and* $[A4]$. *Then, for all* $M > 0$, *as* $nh_n^p \to 0$,

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|\left(\sqrt{n}(\tilde{\theta}_{1,p,n}^{(l_0)} - \theta_1^*), \sqrt{nh_n}(\tilde{\theta}_{2,p,n}^{(k_0)} - \theta_2^*)\right)\right|^M\right] < \infty.$$

Theorem 1. *Let* $p \geq 2$, $l_0 = \left[\frac{p-1}{2}\right]$ *and* $k_0 = \left[\frac{p}{2}\right]$. *Assume* $[A1]$, $[A2](2k_0, 2k_0+1)$, $[A3]$ *and* $[A4]$. *Then, as* $nh_n^p \to 0$,

$$(\sqrt{n}(\tilde{\theta}_{1,p,n}^{(l_0)} - \theta_1^*), \sqrt{nh_n}(\tilde{\theta}_{2,p,n}^{(k_0)} - \theta_2^*)) \xrightarrow{d} (\zeta_1, \zeta_2) \sim N_{p_1+p_2}(0, \mathit{diag}[\Gamma_1(\theta_1^*)^{-1}, \Gamma_2(\theta^*)^{-1}])$$

and

$$E_{\theta^*}[f(\sqrt{n}(\tilde{\theta}_{1,p,n}^{(l_0)} - \theta_1^*), \sqrt{nh_n}(\tilde{\theta}_{2,p,n}^{(k_0)} - \theta_2^*))] \to \mathbb{E}[f(\zeta_1, \zeta_2)]$$

for all continuous functions f *of at most polynomial growth.*

Remark 4. *When $nh_n^p \to 0$ ($p \geq 3$), the adaptive Bayes type estimator is obtained by the initial estimator $\tilde{\theta}_{1,p,n}^{(0)}$ and $U_{p,n}(\theta_1, \theta_2)$. Roughly speaking, in case that $p = 2m + 1$ ($k_0 = m$ and $l_0 = m$),*

$$
\begin{aligned}
\tilde{\theta}_{2,p,n}^{(1)} &\leftarrow U_{p,n}(\tilde{\theta}_{1,p,n}^{(0)}, \theta_2) \\
\tilde{\theta}_{1,p,n}^{(1)} &\leftarrow U_{p,n}(\theta_1, \tilde{\theta}_{2,p,n}^{(1)}) \\
&\vdots \\
&\vdots \\
\tilde{\theta}_{2,p,n}^{(m)} &\leftarrow U_{p,n}(\tilde{\theta}_{1,p,n}^{(m-1)}, \theta_2) \\
\tilde{\theta}_{1,p,n}^{(m)} &\leftarrow U_{p,n}(\theta_1, \tilde{\theta}_{2,p,n}^{(m)})
\end{aligned}
$$

Since $U_{p,n}(\theta_1, \theta_2)$ is a quite complicated function if p is large, the adaptive estimation by using only $U_{p,n}(\theta_1, \theta_2)$ is not so efficient from a computational point of view.

Another adaptive Bayes type estimator

Next, we propose another adaptive Bayes type estimator.

Let $k_0 = \left[\frac{p}{2}\right]$ and $l_0 = \left[\frac{p-1}{2}\right]$.

Set

$$\begin{aligned}
\mathcal{U}_n^{(1)}(\theta_1, \theta_2) &= U_n^{(0)}(\theta_1), \\
\mathcal{U}_n^{(2)}(\theta_1, \theta_2) &= U_n^{(1)}(\theta_1, \theta_2), \\
\mathcal{U}_n^{(p)}(\theta_1, \theta_2) &= U_{p,n}(\theta_1, \theta_2)
\end{aligned}$$

for $p \geq 3$.

Let

$$\begin{aligned}
\mathcal{H}_{p,n}^{(2k-1)}(\theta_1, \theta_2) &= \frac{1}{n^{1-\frac{2k}{p}}}\mathcal{U}_n^{(2k-1)}(\theta_1, \theta_2), \\
\tilde{\mathcal{H}}_{p,n}^{(2k)}(\theta_1, \theta_2) &= \frac{1}{(nh_n)^{1-\frac{2k}{p-1}}}\mathcal{U}_n^{(2k)}(\theta_1, \theta_2)
\end{aligned}$$

for $k = 1, \ldots, l_0$

For $p \geq 2$, the adaptive Bayes type estimators $\tilde{\vartheta}_{1,p,n}^{(2l_0+1)}$ and $\tilde{\vartheta}_{2,p,n}^{(2k_0)}$ are defined as follows: for $k = 1, 2, \ldots, l_0$,

$$\begin{aligned}
\tilde{\vartheta}_{1,p,n}^{(2k-1)} &= \frac{\int_{\Theta_1} \theta_1 \exp\left\{\mathcal{H}_{p,n}^{(2k-1)}(\theta_1, \tilde{\vartheta}_{2,p,n}^{(2k-2)})\right\} \pi_1(\theta_1) d\theta_1}{\int_{\Theta_1} \exp\left\{\mathcal{H}_{p,n}^{(2k-1)}(\theta_1, \tilde{\vartheta}_{2,p,n}^{(2k-2)})\right\} \pi_1(\theta_1) d\theta_1}, \\
\tilde{\vartheta}_{2,p,n}^{(2k)} &= \frac{\int_{\Theta_2} \theta_2 \exp\left\{\tilde{\mathcal{H}}_{p,n}^{(2k)}(\tilde{\vartheta}_{1,p,n}^{(2k-1)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}{\int_{\Theta_2} \exp\left\{\tilde{\mathcal{H}}_{p,n}^{(2k)}(\tilde{\vartheta}_{1,p,n}^{(2k-1)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\vartheta}_{1,p,n}^{(2l_0+1)} &= \frac{\int_{\Theta_1} \theta_1 \exp\left\{\mathcal{U}_n^{(2l_0+1)}(\theta_1, \tilde{\theta}_{2,p,n}^{(2l_0)})\right\} \pi_1(\theta_1) d\theta_1}{\int_{\Theta_1} \exp\left\{\mathcal{U}_n^{(2l_0+1)}(\theta_1, \tilde{\theta}_{2,p,n}^{(2l_0)})\right\} \pi_1(\theta_1) d\theta_1}, \\
\tilde{\vartheta}_{2,p,n}^{(2(l_0+1))} &= \frac{\int_{\Theta_2} \theta_2 \exp\left\{\mathcal{U}_n^{(2(l_0+1))}(\tilde{\theta}_{1,p,n}^{(2l_0+1)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2}{\int_{\Theta_2} \exp\left\{\mathcal{U}_n^{(2(l_0+1))}(\tilde{\theta}_{1,p,n}^{(2l_0+1)}, \theta_2)\right\} \pi_2(\theta_2) d\theta_2},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_{p,n}^{(2k-1)}(\theta_1, \theta_2) &= \frac{1}{n^{1-\frac{2k}{p}}} \mathcal{U}_n^{(2k-1)}(\theta_1, \theta_2), \\
\tilde{\mathcal{H}}_{p,n}^{(2k)}(\theta_1, \theta_2) &= \frac{1}{(nh_n)^{1-\frac{2k}{p-1}}} \mathcal{U}_n^{(2k)}(\theta_1, \theta_2).
\end{aligned}$$

Here we note that $l_0 \leq k_0 \leq l_0 + 1$.

For example, we consider the case when $p = 4$ ($nh_n^4 \to 0$). Noting that $l_0 = 1$ and $k_0 = 2$, we obtain $\tilde{\vartheta}_{1,p,n}^{(2l_0+1)}$ and $\tilde{\vartheta}_{2,p,n}^{(2k_0)}$ as follows.

Step 1: Obtain the initial estimator $\tilde{\vartheta}_{1,p,n}^{(1)}$ of θ_1:

$$\tilde{\vartheta}_{1,p,n}^{(1)} = \frac{\int_{\Theta_1} \theta_1 \exp\left\{\frac{1}{n^{1-\frac{2}{p}}}\mathcal{U}_n^{(1)}(\theta_1)\right\}\pi_1(\theta_1)d\theta_1}{\int_{\Theta_1} \exp\left\{\frac{1}{n^{1-\frac{2}{p}}}\mathcal{U}_n^{(1)}(\theta_1))\right\}\pi_1(\theta_1)d\theta_1}.$$

Step 2: Obtain the 2nd-step adaptive estimator $\tilde{\vartheta}_{2,p,n}^{(2)}$ of θ_2:

$$\tilde{\vartheta}_{2,p,n}^{(2)} = \frac{\int_{\Theta_2} \theta_2 \exp\left\{\frac{1}{(nh_n)^{1-\frac{2}{p-1}}}\mathcal{U}_n^{(2)}(\tilde{\vartheta}_{1,p,n}^{(1)}, \theta_2)\right\}\pi_2(\theta_2)d\theta_2}{\int_{\Theta_2} \exp\left\{\frac{1}{(nh_n)^{1-\frac{2}{p-1}}}\mathcal{U}_n^{(2)}(\tilde{\vartheta}_{1,p,n}^{(1)}, \theta_2)\right\}\pi_2(\theta_2)d\theta_2}.$$

Step 3: Obtain the 3rd-step adaptive estimator $\tilde{\vartheta}_{1,p,n}^{(3)}$ of θ_1:

$$\tilde{\vartheta}_{1,p,n}^{(3)} = \frac{\int_{\Theta_1} \theta_1 \exp\left\{\mathcal{U}_n^{(3)}(\theta_1, \tilde{\vartheta}_{2,p,n}^{(2)})\right\}\pi_1(\theta_1)d\theta_1}{\int_{\Theta_1} \exp\left\{\mathcal{U}_n^{(3)}(\theta_1, \tilde{\vartheta}_{2,p,n}^{(2)})\right\}\pi_1(\theta_1)d\theta_1}.$$

Step 4: Obtain the 4th-step adaptive estimator $\hat{\vartheta}_{2,p,n}^{(4)}$ of θ_2:

$$\tilde{\vartheta}_{2,p,n}^{(4)} = \frac{\int_{\Theta_2} \theta_2 \exp\left\{\mathcal{U}_n^{(4)}(\tilde{\vartheta}_{1,p,n}^{(3)}, \theta_2)\right\}\pi_2(\theta_2)d\theta_2}{\int_{\Theta_2} \exp\left\{\mathcal{U}_n^{(4)}(\tilde{\vartheta}_{1,p,n}^{(3)}, \theta_2)\right\}\pi_2(\theta_2)d\theta_2}.$$

Proposition 5. *Let* $p \geq 2$, $l_0 = \left[\frac{p-1}{2}\right]$ *and* $k_0 = \left[\frac{p}{2}\right]$. *Assume* $[A1]$, $[A2](2k_0, 2k_0+1)$, $[A3]$ *and* $[A4]$. *Then, for all* $M > 0$, *as* $nh_n^p \to 0$,

$$\sup_{n\in\mathbf{N}} E_{\theta^*}\left[\left|(\sqrt{n}(\tilde{\vartheta}_{1,p,n}^{(2l_0+1)} - \theta_1^*), \sqrt{nh_n}(\tilde{\vartheta}_{2,p,n}^{(2k_0)} - \theta_2^*))\right|^M\right] < \infty.$$

Theorem 2. *Let* $p \geq 2$, $l_0 = \left[\frac{p-1}{2}\right]$ *and* $k_0 = \left[\frac{p}{2}\right]$. *Assume* $[A1]$, $[A2](2k_0, 2k_0+1)$, $[A3]$ *and* $[A4]$. *Then, as* $nh_n^p \to 0$,

$$(\sqrt{n}(\tilde{\vartheta}_{1,p,n}^{(2l_0+1)} - \theta_1^*), \sqrt{nh_n}(\tilde{\vartheta}_{2,p,n}^{(2k_0)} - \theta_2^*)) \xrightarrow{d} (\zeta_1, \zeta_2)$$

and

$$E_{\theta^*}[f(\sqrt{n}(\tilde{\vartheta}_{1,p,n}^{(2l_0+1)} - \theta_1^*), \sqrt{nh_n}(\tilde{\vartheta}_{2,p,n}^{(2k_0)} - \theta_2^*))] \to \mathbb{E}[f(\zeta_1, \zeta_2)]$$

for all continuous functions f *of at most polynomial growth.*

Remark 5. *When $nh_n^p \to 0$ ($p \geq 3$), another adaptive Bayes type estimator is obtained by the initial estimator $\tilde{\vartheta}_{1,p,n}^{(1)}$ and $\mathcal{U}_n^{(2)}(\theta_1,\theta_2), U_{3,n}(\theta_1,\theta_2),\ldots,U_{p,n}(\theta_1,\theta_2)$. Roughly speaking, in case that $p = 2m+1$ ($k_0 = m$ and $l_0 = m$),*

Adaptive estimator in Theorem 2

$$
\begin{aligned}
\tilde{\vartheta}_{2,p,n}^{(2)} &\leftarrow \mathcal{U}_n^{(2)}(\tilde{\vartheta}_{1,p,n}^{(1)},\theta_2)\\
\tilde{\vartheta}_{1,p,n}^{(3)} &\leftarrow U_{3,n}(\theta_1,\tilde{\vartheta}_{2,p,n}^{(2)})\\
&\vdots\\
&\vdots\\
\tilde{\vartheta}_{2,p,n}^{(2m)} &\leftarrow U_{2m,n}(\tilde{\vartheta}_{1,p,n}^{(2m-1)},\theta_2)\\
\tilde{\vartheta}_{1,p,n}^{(2m+1)} &\leftarrow U_{2m+1,n}(\theta_1,\tilde{\vartheta}_{2,p,n}^{(2m)})
\end{aligned}
$$

Adaptive estimator in Theorem 1

$$
\begin{aligned}
\tilde{\theta}_{2,p,n}^{(1)} &\leftarrow U_{2m+1,n}(\tilde{\theta}_{1,p,n}^{(0)},\theta_2)\\
\tilde{\theta}_{1,p,n}^{(1)} &\leftarrow U_{2m+1,n}(\theta_1,\tilde{\theta}_{1,p,n}^{(1)})\\
&\vdots\\
&\vdots\\
\tilde{\theta}_{2,p,n}^{(m)} &\leftarrow U_{2m+1,n}(\tilde{\theta}_{1,p,n}^{(m-1)},\theta_2)\\
\tilde{\theta}_{1,p,n}^{(m)} &\leftarrow U_{2m+1,n}(\theta_1,\tilde{\theta}_{2,p,n}^{(m)})
\end{aligned}
$$

Note that $\mathcal{U}_n^{(2)}(\theta_1,\theta_2), U_{3,n}(\theta_1,\theta_2),\ldots,U_{2m,n}(\theta_1,\theta_2)$ have simpler expressions than $U_{2m+1,n}(\theta_1,\theta_2)$.

Compared with the adaptive estimator $(\tilde{\theta}_{1,p,n}^{(l_0)},\tilde{\theta}_{2,p,n}^{(k_0)})$ in Theorem 1, another adaptive estimator $(\tilde{\vartheta}_{1,p,n}^{(2l_0+1)},\tilde{\vartheta}_{2,p,n}^{(2k_0)})$ in Theorem 2 can be efficiently computed from the viewpoint of numerical analysis.

4. Example and simulation result

Consider the two-dimensional diffusion process defined by

$$dX_t = \begin{pmatrix} -0.5X_{t,1} + \beta(\sin X_{t,2} + 2) \\ 15(\cos X_{t,1} + 2) - 0.3X_{t,2} \end{pmatrix} dt + \begin{pmatrix} \alpha & 1 \\ 1 & 4 \end{pmatrix} dw_t, \quad t \in [0,T], \quad X_0 = \begin{pmatrix} 100 \\ 110 \end{pmatrix}, \ (2)$$

where α and β are unknown parameters, $\alpha \in \Theta_1 = [2,12]$ and $\beta \in \Theta_2 = [20,40]$, the true parameter values are $\alpha^* = 5$ and $\beta^* = 30$.

We examine the asymptotic behavior of the adaptive Bayes type estimators with respect to the uniform priors $\pi_1(\alpha)$ and $\pi_2(\beta)$ through the simulations.

Let $a(x,\beta) = \begin{pmatrix} -0.5x_1 + \beta(\sin x_2 + 2) \\ 15(\cos x_1 + 2) - 0.3x_2 \end{pmatrix}$ and $B(x,\alpha) = \begin{pmatrix} \alpha & 1 \\ 1 & 4 \end{pmatrix}^2$.

When we take $p = 2$, the adaptive Bayes type estimators $\tilde{\alpha}^{(0)}_{2,n}$ and $\tilde{\beta}^{(1)}_{2,n}$ are defined as

$$\tilde{\alpha}^{(0)}_{2,n} = \frac{\int_{\Theta_1} \alpha \exp\left\{U^{(0)}_n(\alpha)\right\} \pi_1(\alpha) d\alpha}{\int_{\Theta_1} \exp\left\{U^{(0)}_n(\alpha)\right\} \pi_1(\alpha) d\alpha},$$

$$\tilde{\beta}^{(1)}_{2,n} = \frac{\int_{\Theta_2} \beta \exp\left\{U^{(1)}_n(\tilde{\alpha}^{(0)}_{2,n}, \beta)\right\} \pi_2(\beta) d\beta}{\int_{\Theta_2} \exp\left\{U^{(1)}_n(\tilde{\alpha}^{(0)}_{2,n}, \beta)\right\} \pi_2(\beta) d\beta},$$

where

$$U^{(0)}_n(\alpha) = -\frac{1}{2}\sum_{i=1}^{n}\left\{h_n^{-1} B^{-1}_{i-1}(\alpha)[(\Delta X_i)^{\otimes 2}] + \log\det(B_{i-1}(\alpha))\right\},$$

$$U^{(1)}_n(\alpha, \beta) = -\frac{1}{2}\sum_{i=1}^{n}\left\{h_n^{-1} B^{-1}_{i-1}(\beta)[(\Delta X_i - h_n a_{i-1}(\alpha))^{\otimes 2}] + \log\det(B_{i-1}(\beta))\right\}.$$

Note that $(\sqrt{n}(\tilde{\alpha}^{(0)}_{2,n} - \alpha^*), \sqrt{nh_n}(\tilde{\beta}^{(1)}_{2,n} - \beta))$ has asymptotic normality as $nh_n^2 \to 0$.

On the other hand, if we put $p = 3$, which means that $k_0 = 1$ and $l_0 = 1$, then the adaptive Bayes type estimators $\tilde{\alpha}^{(1)}_{3,n}$ and $\tilde{\beta}^{(1)}_{3,n}$ are defined as

$$\begin{aligned}
\tilde{\alpha}^{(0)}_{3,n} &= \frac{\int_{\Theta_1} \alpha \exp\left\{\frac{1}{n^{1/3}} U_n^{(0)}(\alpha)\right\} \pi_1(\alpha) d\alpha}{\int_{\Theta_1} \exp\left\{\frac{1}{n^{1/3}} U_n^{(0)}(\alpha)\right\} \pi_1(\alpha) d\alpha}. \\
\tilde{\beta}^{(1)}_{3,n} &= \frac{\int_{\Theta_2} \beta \exp\left\{U_{3,n}(\tilde{\alpha}^{(0)}_{3,n}, \beta)\right\} \pi_2(\beta) d\beta}{\int_{\Theta_2} \exp\left\{U_{3,n}(\tilde{\alpha}^{(0)}_{3,n}, \beta)\right\} \pi_2(\beta) d\beta}, \\
\tilde{\alpha}^{(1)}_{3,n} &= \frac{\int_{\Theta_1} \alpha \exp\left\{U_{3,n}(\alpha, \tilde{\beta}^{(1)}_{3,n})\right\} \pi_1(\alpha) d\alpha}{\int_{\Theta_1} \exp\left\{U_{3,n}(\alpha, \tilde{\beta}^{(1)}_{3,n})\right\} \pi_1(\alpha) d\alpha},
\end{aligned}$$

where

$$\begin{aligned}
U_{3,n}(\alpha, \beta) &= -\frac{1}{2} \sum_{i=1}^{n} \left\{ h_n^{-1} \left\{ B_{i-1}^{-1}(\alpha) - h_n B_{i-1}^{-1}(\alpha) \gamma^{(2)}(X_{t^n_{i-1}}, \alpha, \beta) \right\} \left[(\Delta X_i - h_n a_{i-1}(\beta))^{\otimes 2} \right] \right. \\
&\quad \left. + \log \det(B_{i-1}(\alpha)) + h_n tr[B_{i-1}^{-1}(\alpha) \gamma^{(2)}(X_{t^n_{i-1}}, \alpha, \beta)] \right\}, \\
\gamma^{(2)}_{k,l}(x, \alpha, \beta) &= \frac{1}{2} + \sum_{j=1}^{d} \left\{ (\partial_{x_j} a_k(x, \beta)) B_{j,l}(x, \alpha) + (\partial_{x_j} a_l(x, \beta)) B_{j,k}(x, \alpha) \right\}.
\end{aligned}$$

for $k, l = 1, 2$.

Note that $(\sqrt{n}(\tilde{\alpha}^{(1)}_{3,n} - \alpha^*), \sqrt{n h_n}(\tilde{\beta}^{(1)}_{3,n} - \beta))$ has asymptotic normality as $n h_n^3 \to 0$.

Moreover, for the case that $p = 3$, the adaptive Bayes type estimators of Theorem 2, which are denoted by $\bar{\alpha}_{3,n}^{(3)}$ and $\bar{\beta}_{3,n}^{(2)}$, are defined as

$$\begin{aligned}
\bar{\alpha}_{3,n}^{(1)} &= \frac{\int_{\Theta_1} \alpha \exp\left\{\frac{1}{n^{1/3}} U_n^{(0)}(\alpha)\right\} \pi_1(\alpha) d\alpha}{\int_{\Theta_1} \exp\left\{\frac{1}{n^{1/3}} U_n^{(0)}(\alpha)\right\} \pi_1(\alpha) d\alpha}, \\
\bar{\beta}_{3,n}^{(2)} &= \frac{\int_{\Theta_2} \beta \exp\left\{U_n^{(1)}(\bar{\alpha}_{3,n}^{(1)}, \beta)\right\} \pi_2(\beta) d\beta}{\int_{\Theta_2} \exp\left\{U_n^{(1)}(\bar{\alpha}_{3,n}^{(1)}, \beta)\right\} \pi_2(\beta) d\beta}, \\
\bar{\alpha}_{3,n}^{(3)} &= \frac{\int_{\Theta_1} \alpha \exp\left\{U_{3,n}(\alpha, \bar{\beta}_{3,n}^{(2)})\right\} \pi_1(\alpha) d\alpha}{\int_{\Theta_1} \exp\left\{U_{3,n}(\alpha, \bar{\beta}_{3,n}^{(2)})\right\} \pi_1(\alpha) d\alpha}.
\end{aligned}$$

Note that $(\sqrt{n}(\bar{\alpha}_{3,n}^{(3)} - \alpha^*), \sqrt{nh_n}(\bar{\beta}_{3,n}^{(2)} - \beta^*))$ has asymptotic normality as $nh_n^3 \to 0$.

The simulations were done for each $T = 5, 10, 15, 20$ and $h_n = 1/250$.

For the true model, 1000 independent sample paths are generated by the Milstein scheme, and the mean and the standard deviation (s.d.) for the three kinds of estimators are computed and shown in Tables 1 and 2 below.

$$dX_t = \begin{pmatrix} -0.5X_{t,1} + \beta(\sin X_{t,2} + 2) \\ 15(\cos X_{t,1} + 2) - 0.3X_{t,2} \end{pmatrix} dt + \begin{pmatrix} \alpha & 1 \\ 1 & 4 \end{pmatrix} dw_t,\ \ t \in [0, T],\ \ X_0 = \begin{pmatrix} 100 \\ 110 \end{pmatrix}.$$

Table 1: The adaptive Bayes type estimators in Thm. 1 for $p = 2$ and $p = 3$ when $\alpha^* = 5$, $\beta^* = 30$ and $h_n = 1/250$.

	$p = 3$ $(nh_n^3 \to 0)$			$p = 2$ $(nh_n^2 \to 0)$	
T	$\tilde{\alpha}_{3,n}^{(0)}$	$\tilde{\beta}_{3,n}^{(1)}$	$\tilde{\alpha}_{3,n}^{(1)}$	$\tilde{\alpha}_{2,n}^{(0)}$	$\tilde{\beta}_{2,n}^{(1)}$
5	5.24474 (0.11341)	29.74458 (0.96594)	5.02758 (0.11000)	5.19673 (0.11232)	29.91418 (0.96852)
10	5.22368 (0.07942)	29.76372 (0.68014)	5.02580 (0.07664)	5.19302 (0.07893)	29.93529 (0.68185)
15	5.21126 (0.05751)	29.77433 (0.57675)	5.02219 (0.05584)	5.18771 (0.05723)	29.94524 (0.57816)
20	5.20505 (0.04903)	29.77227 (0.49262)	5.02047 (0.04735)	5.18675 (0.04910)	29.95103 (0.49866)

$$dX_t = \begin{pmatrix} -0.5X_{t,1} + \beta(\sin X_{t,2} + 2) \\ 15(\cos X_{t,1} + 2) - 0.3X_{t,2} \end{pmatrix} dt + \begin{pmatrix} \alpha & 1 \\ 1 & 4 \end{pmatrix} dw_t, \ \ t \in [0, T], \ \ X_0 = \begin{pmatrix} 100 \\ 110 \end{pmatrix}.$$

Table 2: The two kinds of adaptive Bayes type estimators for $p = 3$ when $\alpha^* = 5$, $\beta^* = 30$ and $h_n = 1/250$.

	Ada. est. (Thm. 1)		Ada. est. (Thm. 2)	
T	$\tilde{\beta}^{(1)}_{3,n}$	$\tilde{\alpha}^{(1)}_{3,n}$	$\bar{\beta}^{(2)}_{3,n}$	$\bar{\alpha}^{(3)}_{3,n}$
5	29.74458 (0.96594)	5.02758 (0.11000)	29.88470 (1.02515)	5.02550 (0.10568)
10	29.76372 (0.68014)	5.02580 (0.07664)	29.92011 (0.70709)	5.02331 (0.07253)
15	29.77433 (0.57675)	5.02219 (0.05584)	29.92970 (0.58403)	5.02284 (0.06037)
20	29.77227 (0.49262)	5.02047 (0.04735)	29.93584 (0.49174)	5.02156 (0.05270)