Z-process method for statistical change point problems

Ilia Negri

University of Bergamo (Italy)

(Joint work with Y. Nishiyama)

SAPS – X, Le Mans
March 17–20, 2015

Outline of the talk

Overview

- Testing on structural change problems has been an important issue in statistics
- Horváth and Parzen (1994) are apparently the firsts to introduce a statistics based on the Fisher-score process to test a parameter change for independent data
- N & N (2012) take the same approach to the change point problem for an ergodic diffusion process model based on the continuous observation and also the consistency of the test under an alternative which has sufficient generality was proved
- This work is an attempt to present a general and unified approach for change point problems. It is based on the partial sum process of estimating equations.

Preliminaries

Let us give an illustration by the example of independent data:

- Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space
- Let $f(\cdot;\theta)$ a parametric family of probability densities with respect to μ, where $\theta \in \Theta \subset \mathbb{R}^d$.
- Let $X_1, X_2, \ldots$ be an independent sequence of $\mathcal{X}$-valued random variables from this parametric model.

Define

$$\theta \mapsto \mathbb{M}_n(\theta) = \sum_{k=1}^{n} \log f(X_k;\theta),$$

and

$$\mathbb{Z}_n(\theta) = \dot{\mathbb{M}}_n(\theta)$$

where $\dot{\mathbb{M}}_n(\theta)$ is the gradient vector of $\mathbb{M}_n(\theta)$.

Test for change point in i.i.d. case

Suppose we consider the following testing problem (change point problem):

H_0: the true value $\theta_0 \in \Theta$ does not change during $u \in [0,1]$

versus any alternative that we can state as H_1: "not H_0".
To deal with this change point problems let us introduce the partial sum process

$$\mathbb{M}_n(u,\theta) = \sum_{k=1}^{[un]} \log f(X_k;\theta), \quad \forall u \in [0,1],$$

and denote its gradient vectors by

$$\mathbb{Z}_n(u,\theta) = \dot{\mathbb{M}}_n(u,\theta)$$

Horváth and Parzen (1994) statistics

Let $\widehat{\theta}_n$ be the MLE for the full data $X_1, \ldots, X_n$ as a special case of Z-estimators, that is, $\widehat{\theta}_n$ is the solution to the estimating equation

$$\mathbb{Z}_n(1, \theta) = \dot{\mathbb{M}}_n(1, \theta) = 0.$$

To test a change in the value of the parameter, Horváth and Parzen (1994) introduce the test statistic

$$\mathcal{T}_n = n^{-1} \sup_{u \in [0,1]} \mathbb{Z}_n(u, \widehat{\theta}_n)^\top \widehat{I}_n^{-1} \mathbb{Z}_n(u, \widehat{\theta}_n),$$

where $\widehat{I}_n$ is a consistent estimator for the Fisher Information matrix $I(\theta_0)$.

Limit distribution

- From Donsker's theorem

$$u \rightsquigarrow \sqrt{n}\mathbb{Z}_n(u, \theta_0) \quad \text{converges weakly to} \quad u \rightsquigarrow I(\theta_0)^{1/2}B(u)$$

in the Skorohod space $D[0,1]$, where $u \rightsquigarrow B(u)$ is a vector of independent standard Brownian motions.

- It also holds

$$u \rightsquigarrow \sqrt{n}\mathbb{Z}_n(u, \widehat{\theta}_n) \quad \text{converges weakly to} \quad u \rightsquigarrow I(\theta_0)^{1/2}B^\circ(u)$$

in $D[0,1]$, where $u \rightsquigarrow B^\circ(u)$ is a vector of independent standard Brownian bridges.

- From the continuous mapping theorem

$$\mathcal{T}_n \to^d \sup_{u \in [0,1]} ||B^\circ(u)||^2.$$

Beyond the Horváth and Parzen approach

- Horváth and Parzen did not discuss the asymptotic behaviour of the test statistics under the alternative.
- The same approach of Horváth and Parzen taken by N.N to change point problem for ergodic diffusion process with continuous observations can be applied to prove consistency of the test in i.i.d model.
- In this work we present a generalised version of this method which works also for some more general models.
- The method presented seems not just a simple generalisation of the Fisher score process proposed by Horváth and Parzen for i.i.d model but make possible to treat statistical change point problem for more general model including non ergodic cases.

Set up of the change point problem

All process $u \rightsquigarrow X(u)$, are assumed to take values in $D[0,1]$, the space of cad lag functions defined on $[0,1]$ equipped with the Skorohod metric.

- Let $\Theta \subset \mathbb{R}^d$ be bounded, open and convex
- Let $u \rightsquigarrow \mathbb{Z}_n(u,\theta)$ be an $\mathbb{R}^d$-valued random process indexed by $\theta \in \Theta \subset \mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{F}, P)$ that is common for all $n \in \mathbb{N}$
- This framework includes the case where $\mathbb{Z}_n(u,\theta) = \dot{\mathbb{M}}_n(u,\theta)$ where $\theta \mapsto \mathbb{M}_n(u,\theta)$ is an $\mathbb{R}$-valued random field which is assumed to be two times continuously differentiable with the Hessian matrix $\ddot{\mathbb{M}}_n(u,\theta)$.

We consider the following testing problem:
H_0: the true value $\theta_0 \in \Theta$ does not change during $u \in [0,1]$;
H_1: "not H_0".

Conditions under H_0 for the limit process

Under H_0, let be given the *limit* process $u \rightsquigarrow Z_{\theta_0}(u, \theta)$ and a sequence of diagonal matrices Q_n such that

1. It holds that

$$\sup_{\theta \in \Theta} ||Q_n^{-2} \mathbb{Z}_n(1, \theta) - Z_{\theta_0}(1, \theta)|| \to^p 0$$

2. The limits $Z_{\theta_0}(1, \theta)$ satisfy

$$\inf_{\theta : ||\theta - \theta_0|| > \varepsilon} ||Z_{\theta_0}(1, \theta)|| > 0 = ||Z_{\theta_0}(1, \theta_0)||,$$

almost surely, $\forall \varepsilon > 0$

Conditions under H_0 for the limit process (continue)

3. There exist a sequence of diagonal matrices R_n and a sequence of matrix valued random processes $u \rightsquigarrow V_n(u, \theta_0)$ such that $V_n(1, \theta_0)$'s are regular almost surely and that for any sequence of Θ-valued random vectors $\widetilde{\theta}_n(u)$ indexed by $u \in [0, 1]$ satisfying $\sup_{u \in [0,1]} ||\widetilde{\theta}_n(u) - \theta_0|| \to^p 0$,

$$\sup_{u \in [0,1]} ||Q_n^{-1} \dot{\mathbb{Z}}_n(u, \widetilde{\theta}_n(u)) R_n^{-1} - (-V_n(u, \theta_0))|| \to^p 0.$$

4. Moreover in $D[0, 1]$

$$(Q_n^{-1} \mathbb{Z}_n(u, \theta_0), V_n(u, \theta_0)) \to^d ((u^{-1} V(u, \theta_0))^{1/2} B(u), V(u, \theta_0)),$$

where $u \rightsquigarrow B(u)$ is a d-dimensional standard Brownian motion

The test statistics

- Let $\widehat{\theta}_n$ be any sequence of Θ-valued random vectors such that $||Q_n^{-1}\mathbb{Z}_n(1,\widehat{\theta}_n)|| = o_P(1)$ under H_0
- Let $u \rightsquigarrow V(u,\theta_0)$ be a non-negative definite matrix valued random process such that $V(1,\theta_0)$ is positive definite almost surely, and let $u \rightsquigarrow B(u)$ be a vector of independent standard Brownian motions;
- Let $u \rightsquigarrow \widehat{V}_n(u)$ a uniformly consistent sequence of estimators for the non-negative definite matrix valued random process $u \rightsquigarrow V(u,\theta_0)$

Introduce the test statistic

$$\mathcal{T}_n = \sup_{u\in(0,1]} (Q_n^{-1}\mathbb{Z}_n(u,\widehat{\theta}_n))^\top (u\widehat{V}_n(u)^{-1}) Q_n^{-1}\mathbb{Z}_n(u,\widehat{\theta}_n).$$

The main result under H_0

Theorem

Under H_0 and related above conditions, if

$$\sup_{u\in[0,1]} ||\widehat{V}_n(u) - V(u,\theta_0)|| \to^p 0,$$

then it holds that

$$\mathcal{T}_n \to^d \sup_{u\in[0,1]} ||B(u) - u^{1/2}V(u,\theta_0)^{1/2}V(1,\theta_0)^{-1/2}B(1)||^2.$$

Remarks

- The underlying probability spaces do not have to be common for $n \in \mathbb{N}$ if $V(u, \theta_0)$ appearing in the limit is non-random.
- In the examples the rate matrices Q_n and R_n are diagonal matrices like $\sqrt{n} I_d$, and they are given such that $Q_n^{-1} \ddot{\mathbb{M}}_n(u, \theta) R_n^{-1}$ converges to a limit, where $\ddot{\mathbb{M}}_n(u, \theta)$ is a Hessain matrix of the partial sum process $\mathbb{M}_n(u, \theta)$ of a contrast function like a loglikelihood function.
- If $V(u, \theta_0) = uV(1, \theta_0)$ for every $u \in [0, 1]$, then the test is asymptotically distribution free. In this case $\mathcal{T}_n \to^d \sup_{u \in [0,1]} ||B^\circ(u)||^2$ where $u \rightsquigarrow B^\circ(u) = B(u) - uB(1)$ is a vector of independent standard Brownian bridges.
- In the general case, if $u \rightsquigarrow V(u, \theta_0)$ and $u \rightsquigarrow B(u)$ are independent, then the limit is approximated by

$$\sup_{u \in [0,1]} ||B(u) - u^{1/2} \widehat{V}_n(u)^{1/2} \widehat{V}_n(1)^{-1/2} B(1)||^2$$

Conditions under H_1 for the limit process

Under H_1, let be given the *limit* process $u \rightsquigarrow \mathcal{Z}(u,\theta)$ such that:

1. there exists a sequence of diagonal matrices Q_n such that

$$\sup_{u\in[0,1]} \sup_{\theta\in\Theta} ||Q_n^{-2}\mathbb{Z}_n(u,\theta) - \mathcal{Z}(u,\theta)|| \to^p 0,$$

2. There exists a vector $\theta_* \in \Theta$ such that

$$\inf_{\theta:||\theta-\theta_*||>\varepsilon} ||\mathcal{Z}(1,\theta)|| > 0 = ||\mathcal{Z}(1,\theta_*)||$$

almost surely, $\forall\varepsilon > 0$

3. For the same θ_* it holds that

$$\sup_{u\in(0,1)} ||\mathcal{Z}(u,\theta_*)|| > 0, \quad \text{almost surely.}$$

Remarks (1)

- Assuming conditions (1) and (2) under H_1 is natural. See e.g. Theorems 5.7 and 5.9 of van der Vaart (1998).
- Condition (3) under H_1 can be checked easily when the alternatives has the form:
  H_1': there exists a constant $u_* \in (0,1)$ such that the true value is $\theta_0 \in \Theta$ for $u \in [0, u_*]$, and $\theta_1 \in \Theta$ for $u \in (u_*, 1]$, where $\theta_0 \neq \theta_1$.
- Condition (1) under H_1' is satisfied with $\mathcal{Z}(u, \theta)$ such that

$$\mathcal{Z}(u_*, \theta) = u_* Z_{\theta_0}(1, \theta)$$

and

$$\mathcal{Z}(1, \theta) = u_* Z_{\theta_0}(1, \theta) + (1 - u_*) Z_{\theta_1}(1, \theta)$$

where $Z_{\theta_1}(1, \theta)$ are also assumed to satisfy (2) under H_0

Remarks (2)

- Condition (3) is satisfied observing that

$$\mathcal{Z}(u_*, \theta_*) = u_*(1 - u_*)(Z_{\theta_0}(1, \theta_*) - Z_{\theta_1}(1, \theta_*));$$

  is greater than 0 with probability one.
- If this were zero with positive probability, then it should follow from $\mathcal{Z}(1, \theta_*) = 0$ that $Z_{\theta_0}(1, \theta_*) = Z_{\theta_1}(1, \theta_*) = 0$ with positive probability, and this contradicts (2) under H_0 and the assumption that $\theta_0 \neq \theta_1$
- Hence, it holds, almost surely,

$$\begin{aligned} \sup_{u \in (0,1)} ||\mathcal{Z}(u, \theta_*)|| &\geq ||\mathcal{Z}(u_*, \theta_*)|| \\ &= u_*(1 - u_*)||Z_{\theta_0}(1, \theta_*) - Z_{\theta_1}(1, \theta_*)|| > 0 \end{aligned}$$

- This positive value is closely related to the power of our test under H_1'

The main result under H_1

Theorem

Under H_1 and related above conditions, it holds for any random point $\check{u}$ in $(0,1)$ that

$$\mathcal{T}_n \geq \lambda(\check{u} Q_n \widehat{V}_n(\check{u})^{-1} Q_n) \left\{ ||\mathcal{Z}(\check{u},\theta_*)||^2 + o_P(1) \right\},$$

where $\lambda(A)$ denotes the smallest eigenvalue of the random matrix A. Hence, if there exists a random point $\check{u}$ in $(0,1)$ such that

$$||\mathcal{Z}(\check{u},\theta_*)|| > 0$$

almost surely, and such that

$$\lambda(Q_n(\widehat{V}_n(\check{u})^{-1} Q_n) \to^p \infty$$

then the test is consistent.

Consistency of $\mathbb{Z}$ estimators

The following Lemmas prove the consistency of a sequence of Z-estimators. It can be proved exactly in the same way as Theorems 5.7 and 5.9 of van der Vaart (1998),

Lemma

Let conditions under H_0 be satisfied. Then for any sequence of Θ-valued random vectors $\widehat{\theta}_n$ such that $||\mathbb{Z}_n(1, \widehat{\theta}_n)|| = o_P(1)$, it holds that $\widehat{\theta}_n \to^p \theta_0$.

Lemma

Let conditions under H_1 be satisfied. Then for any sequence of Θ-valued random vectors $\widehat{\theta}_n$ such that $||\mathbb{Z}_n(1, \widehat{\theta}_n)|| = o_P(1)$, it holds that $\widehat{\theta}_n \to^p \theta_*$.

Ergodic diffusion process: set up

Let us consider an $I = (l, r)$-valued diffusion process $t \rightsquigarrow X_t$ unique strong solution solution to the stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t S(X_s; \alpha) ds + \int_0^t \sigma(X_s; \beta) dW_s,$$

where $s \rightsquigarrow W_s$ is a standard Wiener process.

- The parameter is $\theta = (\alpha^\top, \beta^\top)^\top$ where $\alpha \in \Theta_A \subset \mathbb{R}^{d_A}$ and $\beta \in \Theta_B \subset \mathbb{R}^{d_B}$.

Ergodic diffusion process: sample scheme

We observe the process X at discrete time grids $0 = t_0^n < t_1^n < \cdots < t_n^n$, and the asymptotic scheme is $n\Delta_n^2 \to 0$ and $t_n^n \to \infty$ as $n \to \infty$, where

$$\Delta_n = \max_{1 \le k \le n} |t_k^n - t_{k-1}^n|,$$

and

$$\sum_{k=1}^{n} \left| \frac{|t_k^n - t_{k-1}^n|}{t_n^n} - \frac{1}{n} \right| \to 0, \quad \text{as } n \to \infty. \tag{1}$$

Ergodic diffusion process: $\mathbb{Z}$ process

We introduce the $(d_A + d_B)$-dimensional random vectors

$$\mathbb{Z}_n(u,\theta) = \dot{\mathbb{M}}_n(u,\theta) = \dot{\gamma}_n(u,\theta),$$

where

$$\gamma_n(u,\theta) = -\sum_{k: t_{k-1}^n \leq u t_n^n} \left\{ \log \sigma(X_{t_{k-1}^n};\beta) + \frac{|X_{t_k^n} - X_{t_{k-1}^n} - S(X_{t_{k-1}^n};\alpha)|t_k^n - t_{k-1}^n||^2}{2\sigma(X_{t_{k-1}^n};\beta)^2 |t_k^n - t_{k-1}^n|} \right\}$$

and $R_n = Q_n$ are the diagonal matrix such that $R_n^{(i,i)}$ is $\sqrt{t_n^n}$ for $i = 1, ..., d_A$ and $\sqrt{n}$ for $i = d_A + 1, ..., d$ with $d = d_A + d_B$.

Ergodic diffusion process: $\mathbb{Z}$ process

We re-write with natural notation, the $(d_A + d_B)$-dimensional random vectors

$$\mathbb{Z}_n(u,\theta) = \dot{\mathbb{M}}_n(u,\theta) = (\dot{\mathbb{M}}_n^A(u,\theta)^\top, \dot{\mathbb{M}}_n^B(u,\theta)^\top)^\top,$$

We consider the $(d_A + d_B) \times (d_A + d_B)$-random matrices

$$\dot{\mathbb{Z}}_n(u,\theta) = \ddot{\mathbb{M}}_n(u,\theta) = \left(\begin{array}{cc} \ddot{\mathbb{M}}_n^A(u,\theta) & \ddot{\mathbb{M}}_n^C(u,\theta) \\ \ddot{\mathbb{M}}_n^C(u,\theta)^\top & \ddot{\mathbb{M}}_n^B(u,\theta) \end{array} \right).$$

Ergodic diffusion process: $\mathbb{Z}$ process

Under some regularity conditions it can be proved that

- $$\sup_{u\in[0,1]}\left\|\frac{1}{t_n^n}\dot{\mathbb{M}}_n^A(u,\theta_0)-\frac{1}{t_n^n}\sum_{k:t_{k-1}^n\le ut_n^n}\frac{\dot{S}(X_{t_{k-1}^n};\alpha_0)}{\sigma(X_{t_{k-1}^n};\beta_0)}(W_{t_k^n}-W_{t_{k-1}^n})\right\|=o_P((t_n^n)^{-1/2})$$

- $$\sup_{u\in[0,1]}\left\|\frac{1}{n}\dot{\mathbb{M}}_n^B(u,\theta_0)-\frac{1}{n}\sum_{k:t_{k-1}^n\le ut_n^n}\frac{\dot{\sigma}(X_{t_{k-1}^n};\beta_0)}{\sigma(X_{t_{k-1}^n};\beta_0)}\left\{\frac{|W_{t_k^n}-W_{t_{k-1}^n}|^2}{|t_k^n-t_{k-1}^n|}-1\right\}\right\|=o_P(n^{-1/2})$$

- $$\sup_{u\in[0,1]}\sup_{\theta\in\Theta}\left\|\frac{1}{t_n^n}\ddot{\mathbb{M}}_n^A(u,\theta)-\frac{1}{t_n^n}\sum_{k:t_{k-1}^n\le ut_n^n}H^A(X_{t_{k-1}^n};\theta_0,\theta)|t_k^n-t_{k-1}^n|\right\|=o_P((t_n^n)^{-1/2})$$

- $$\sup_{u\in[0,1]}\sup_{\theta\in\Theta}\left\|\frac{1}{n}\ddot{\mathbb{M}}_n^B(u,\theta)-\frac{1}{n}\sum_{k:t_{k-1}^n\le ut_n^n}H^B(X_{t_{k-1}^n};\theta_0,\theta)\right\|=o_P(n^{-1/2})$$

- $$\sup_{u\in[0,1]}\sup_{\theta\in\Theta}\left\|\frac{1}{\sqrt{t_n^n n}}\ddot{\mathbb{M}}_n^C(u,\theta)\right\|=o_P(n^{-1/4})$$

Ergodic diffusion process: $\mathbb{Z}$ process

Where

$$\begin{aligned} H^A(x;\theta_0,\theta) &= \frac{\ddot{S}(x;\alpha)(S(x;\alpha_0)-S(x;\alpha))-\dot{S}(x;\alpha)\dot{S}(x;\alpha)^\top}{\sigma(x;\beta)^2}, \\ H^B(x;\theta_0,\theta) &= \left\{\frac{\ddot{\sigma}(x;\beta)}{\sigma(x;\beta)^3}-3\frac{\dot{\sigma}(x;\beta)\dot{\sigma}(x;\beta)^\top}{\sigma(x;\beta)^4}\right\}(\sigma(x;\beta_0)^2-\sigma(x;\beta)^2) \\ &\quad -2\frac{\dot{\sigma}(x;\beta)\dot{\sigma}(x;\beta)^\top}{\sigma(x;\beta)^2}. \end{aligned}$$

Condition (1) under H_0 is satisfied with $\dot{M}_{\theta_0}(1,\theta)=(\dot{M}^A_{\theta_0}(1,\theta)^\top,\dot{M}^B_{\theta_0}(1,\theta)^\top)^\top$, where

$$\dot{M}^A_{\theta_0}(1,\theta)=\int_I\frac{\dot{S}(x;\alpha)}{\sigma(x;\beta)}(S(x;\alpha_0)-S(x;\alpha))\mu_{\theta_0}(dx)$$

and

$$\dot{M}^B_{\theta_0}(1,\theta)=\int_I\frac{\dot{\sigma}(x;\beta)}{\sigma(x;\beta)^3}(\sigma(x;\beta_0)^2-\sigma(x;\beta)^2)\mu_{\theta_0}(dx),$$

Ergodic diffusion process: $\mathbb{Z}$ process

The Main Theorem under H_0 holds for

$$V_n(u,\theta_0) = \begin{pmatrix} V_n^A(u,\theta_0) & 0 \\ 0 & V_n^B(u,\theta_0) \end{pmatrix},$$

where

$$\begin{aligned} V_n^A(u,\theta_0) &= \frac{1}{t_n^n} \sum_{k:t_{k-1}^n \le u t_n^n} \frac{\dot S(X_{t_{k-1}^n};\alpha_0)\dot S(X_{t_{k-1}^n};\alpha_0)^\top}{\sigma(X_{t_{k-1}^n};\beta_0)^2} |t_k^n - t_{k-1}^n|, \\ V_n^B(u,\theta_0) &= \frac{2}{n} \sum_{k:t_{k-1}^n \le u t_n^n} \frac{\dot\sigma(X_{t_{k-1}^n};\beta_0)\dot\sigma(X_{t_{k-1}^n};\beta_0)^\top}{\sigma(X_{t_{k-1}^n};\beta_0)^2}. \end{aligned}$$

Ergodic diffusion process: limit process

The limit of $V_n(u,\theta_0)$ is $V(u,\theta_0) = uI_{\theta_0}(\theta_0)$, where

$$I_{\theta_0}(\theta) = \begin{pmatrix} I^A_{\theta_0}(\theta) & 0 \\ 0 & I^B_{\theta_0}(\theta) \end{pmatrix}$$

with

$$I^A_{\theta_0}(\theta) = \int_I \frac{\dot S(x;\alpha)\dot S(x;\alpha)^\top}{\sigma(x;\beta)^2}\mu_{\theta_0}(dx),$$

and

$$I^B_{\theta_0}(\theta) = 2\int_I \frac{\dot\sigma(x;\beta)\dot\sigma(x;\beta)^\top}{\sigma(x;\beta)^2}\mu_{\theta_0}(dx).$$

We suppose that $I_{\theta_0}(\theta)$'s are positive definite.

Ergodic diffusion process: consistent estimator

As a consistent estimator $\widehat{V}_n(u)$ for $V(u, \theta_0)$, we introduce

$$\widehat{V}_n(u) = \begin{pmatrix} u\widehat{I}_n^A & 0 \\ 0 & u\widehat{I}_n^B \end{pmatrix},$$

where

$$\widehat{I}_n^A = \frac{1}{n}\sum_{k=1}^{n} \frac{\dot{S}(X_{t_{k-1}^n}; \widehat{\alpha}_n)\dot{S}(X_{t_{k-1}^n}; \widehat{\alpha}_n)^\top}{\sigma(X_{t_{k-1}^n}; \widehat{\beta}_n)^2},$$

and

$$\widehat{I}_n^B = \frac{2}{n}\sum_{k=1}^{n} \frac{\dot{\sigma}(X_{t_{k-1}^n}; \widehat{\beta}_n)\dot{\sigma}(X_{t_{k-1}^n}; \widehat{\beta}_n)^\top}{\sigma(X_{t_{k-1}^n}; \widehat{\beta}_n)^2}.$$

Ergodic diffusion process: test's consistency

Condition (1) under H_1' is satisfied with

$$\mathcal{Z}(u,\theta) = (u \wedge u_*)\dot{M}_{\theta_0}(1,\theta) + ((u-u_*)\vee 0)\dot{M}_{\theta_1}(1,\theta).$$

Condition (3) under H_1' is automatically satisfied as soon as the natural conditions (2) under H_0 and (2) under H_1' are satisfied.
Under H_1', since the matrix

$$u_* I_{\theta_0}(\theta_*) + (1-u_*)I_{\theta_1}(\theta_*)$$

is positive definite, and

$$\widehat{V}_n(u_*) \to u_* I_{\theta_0}(\theta_*) + (1-u_*)I_{\theta_1}(\theta_*)$$

then

$$\lambda(Q_n \widehat{V}_n(u_*) Q_n) \to^p \infty$$

Hence the test is consistent.

Ergodic diffusion process: numerical results

We consider the Ornstein-Uhlenbeck process starting from $x_0 = 0$ for the true (data-generating) process:

$$X_t = x_0 - \int_0^t \alpha X_s ds + \beta W_t, \quad t \in [0, T].$$

- We treat the equidistant sampling case, that is, $\Delta_n = |t_k^n - t_{k-1}^n|$ for every $k = 1, ..., n$.
- We observe the trajectory of the process for different time horizons $t_n^n = T$
- The number n of observations for each trajectory is such that $t_n^n = n^{1/3}$, so $\Delta_n = n^{-2/3}$.
- We simulate $M = 10^4$ trajectories

The limit distribution of test statistics

- For any fixed level $\varepsilon > 0$ the critical value c_ε is given by

$$P\left(\sup_{u\in[0,1]} \sum_{i=1}^{d_A+d_B} |B^{\circ,(i)}(u)|^2 > c_\varepsilon\right) = \varepsilon.$$

- Table 1 of Lee *et al.* (2003) gives a table of the critical values for the significance levels $\varepsilon = 0.01, 0.05, 0.10$ and for different values of the dimension $d = d_A + d_B$ computed by Monte Carlo simulation for the limit distribution.
- Throughout we take the significance level to be $\varepsilon = 0.05$. For two parameters ($d = 2$) the critical value is $c_\varepsilon = 2.408$.

H_0 true

Empirical size for different time horizons.

T	5	10	15	20	25
n	125	1000	3375	8000	15625
$\alpha_0 = 1,\ \beta_0 = 1$	0.044	0.054	0.050	0.052	0.053
$\alpha_0 = 0.25,\ \beta_0 = 0.02$	0.047	0.061	0.058	0.064	0.054

The empirical size gains along with increasing terminal time $T = t_n^n$, attaining at 0.05, but also for small terminal T. In the second example, the values of the parameter are the maximum likelihood estimate for the mostly federal funds data 1963-1998 in Aït-Sahalia (1999).

H_0 not true

Regarding the alternative hypothesis we study the behavior of the test statistic in three different situations and for different change point $u_* T$ of the parameters, as follows:

- The drift coefficient changes from α_0 to α_1, but the diffusion coefficient does not change.
- The drift coefficient does not change, but the diffusion coefficient changes from β_0 to β_1.
- Both coefficients change.

For each of the above scenarios we consider the following change points, $u_* = \frac{1}{2}, \frac{3}{4}, \frac{9}{10}$.

H_0 not true: 1st scenario, case a

The values of the parameter are $\alpha_0 = 0.25$, $\alpha_1 = 0.50$ and $\beta = 0.02$ (it does not vary).

T n	5 125	10 1000	15 3375	20 8000	25 15625
$u_* = \frac{1}{2}$	0.31	0.52	0.73	0.79	0.88
$u_* = \frac{3}{4}$	0.12	0.17	0.23	0.26	0.35
$u_* = \frac{9}{10}$	0.05	0.07	0.08	0.08	0.09

H_0 not true: 1st scenario, case b

The values of the parameter are $\alpha_0 = 0.25$, $\alpha_1 = 1.25$ and $\beta = 0.02$ (it does not vary).

T / n	5 / 125	10 / 1000	15 / 3375	20 / 8000	25 / 15625
$u_* = \frac{1}{2}$	0.35	0.60	0.78	0.88	0.94
$u_* = \frac{3}{4}$	0.13	0.20	0.28	0.31	0.38
$u_* = \frac{9}{10}$	0.06	0.08	0.09	0.11	0.11

H_0 not true: 2nd scenario, case a

The values of the parameter are $\alpha = 0.25$ (it does not vary), $\beta_0 = 0.02$ and $\beta_1 = 0.03$.

T n	5 125	10 1000	15 3375	20 8000	25 15625
$u_* = \frac{1}{2}$	0.99	1	1	1	1
$u_* = \frac{3}{4}$	0.86	1	1	1	1
$u_* = \frac{9}{10}$	0.36	0.99	1	1	1

H_0 not true: 2nd scenario, case b

The values of the parameter are $\alpha = 0.25$ (it does not vary), $\beta_0 = 0.020$ and $\beta_1 = 0.025$.

T / n	5 / 125	10 / 1000	15 / 3375	20 / 8000	25 / 15625
$u_* = \frac{1}{2}$	0.87	1	1	1	1
$u_* = \frac{3}{4}$	0.52	0.99	1	1	1
$u_* = \frac{9}{10}$	0.14	0.62	0.99	1	1

Volatility of diffusion process: set up

Let us consider an $I = (l, r)$-valued diffusion process $t \rightsquigarrow X_t$ unique strong solution solution to the SDE

$$X_t = X_0 + \int_0^t S(X_s)ds + \int_0^t \sigma(X_s; \theta)dW_s,$$

where $s \rightsquigarrow W_s$ is a standard Wiener process.

- The drift coefficient $S(\cdot)$ is treated as an unknown nuisance function
- The parameter is $\theta \in \Theta \subset \mathbb{R}^d$.

Ergodic diffusion process: sample scheme

We observe the process X at discrete time grids :

$$0 = t_0^n < t_1^n < \cdots < t_n^n = T < \infty$$

and the asymptotic scheme is

$$\sum_{k=1}^{n} \left| \frac{|t_k^n - t_{k-1}^n|}{t_n^n} - \frac{1}{n} \right| \to 0, \quad \text{as } n \to \infty.$$

Volatility of diffusion process: $\mathbb{Z}$ process

We introduce

$$\mathbb{Z}_n(u,\theta) = \dot{\mathbb{M}}_n(u,\theta) = \dot{\gamma}_n(u,\theta)$$

where

$$\gamma_n(u,\theta) = -\sum_{k:t_{k-1}^n \le ut_n^n} \left\{ \log \sigma(X_{t_{k-1}^n};\theta) + \frac{|X_{t_k^n} - X_{t_{k-1}^n}|^2}{2\sigma(X_{t_{k-1}^n};\theta)^2 |t_k^n - t_{k-1}^n|} \right\}.$$

The rate matrices are given by $R_n = Q_n = \sqrt{n} I_d$.
An interesting point of this example is that the limit of $-\ddot{\mathbb{M}}_n(u, \widetilde{\theta}_n(u))$ is random and depend on $u \in [0,1]$ in a complex way.

Volatility of diffusion process: numerical results

The data-generating process is the following:

$$X_t = 4 - \int_0^t (X_s - 4)ds + \int_0^t \exp\left(\theta \frac{X_s^2}{1+X_s^2}\right) dW_s, \quad t \in [0,1],$$

- The sample scheme is the equidistant time grid $t_k^n = \frac{k}{n}$, $k = 0, 1, ..., n$.
- $M = 10^4$ trajectories are simulated
- The test statistic can be computed, estimating

$$V(u, \theta_0) = 2\int_0^u \left| \frac{X_s^2}{1+X_s^2} \right|^2 ds$$

by the natural estimator

$$\widehat{V}_n(u) = \frac{2}{n} \sum_{k=1}^{[un]} \left| \frac{X_{t_{k-1}^n}^2}{1+X_{t_{k-1}^n}^2} \right|^2 .$$

H_0 true

Empirical size for different n. The critical value is $c_\varepsilon = 1.820$.
The true value of the parameter is set as $\theta_0 = 1.0$ or 1.5.

n	20	40	100	200
$\theta_0 = 1.0$	0.013	0.024	0.031	0.034
$\theta_0 = 1.5$	0.028	0.029	0.035	0.036

H_0 not true

The empirical power under H_1'.
The true values of the parameter change from $\theta_0 = 1.0$ to $\theta_1 = 1.5$ at time point $u_* = \frac{1}{2}, \frac{3}{4}$ or $\frac{9}{10}$.

n	20	40	100	200
$u_* = \frac{1}{2}$	0.288	0.793	0.974	0.994
$u_* = \frac{3}{4}$	0.457	0.768	0.951	0.979
$u_* = \frac{9}{10}$	0.251	0.466	0.805	0.935

What else?

Thank you for your attention !!

References

1. Horváth, L. and Parzen, E. (1994). Limit theorems for Fisher-score change processes. IMS Lecture Notes
2. N & N (2012). Asymptotically distribution free test for parameter change in a diffusion process model. *Ann. Inst. Statist. Math.*
3. N & N (2014). Z-process method for change point problems. submittted (and rejected!)