On the invariant measure for branching diffusions with immigrations.

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Finite systems of particles in $\mathbb{R}^d$.

Models biological phenomena, for example dispersion of pollen in space (e.g. genetically modified maize, see Catherine Larédo)

Model assumptions

- Particle i moves according to SDE

$$dX_t^i = b(X_t^i)dt + \sigma(X_t^i)dW_t^i.$$

W^i independent Brownian motions, b, σ smooth.

- Branching :

Particle i in position x at time t dies in $(t, t+h]$ with probability

$$\kappa(x) \cdot h + o(h), \quad h \to 0.$$

$\kappa(.)$: branching rate, smooth, $\kappa : \mathbb{R}^d \to [a, b],\ 0 < a < b < \infty$.

Just before death, particle in position x creates k offspring particles, $k \geq 0$, with probability

$$p_k(x), k \geq 0.$$

$p_k(.)$: offspring law.

- Immigrations :

New particles immigrate at rate $c > 0$ into a random position

$$\sim g(x)dx \text{ on } \mathbb{R}^d,$$

independently of the already existing particles.

g : immigration density.

Observations : continuous time, process of particle configurations :

$$\varphi = (\varphi_t)_{t \geq 0},$$

taking values in the configuration space

$$S := \bigcup_{l \geq 0} (\mathbb{R}^d)^l,$$

càdlàg, strong Markov.

Asymptotic regime

Hypotheses : strictly sub critical branching:

$$\sum kp_k(x) \le \eta, \quad \eta < 1.$$

This implies:

φ ergodic, $\varphi = \emptyset$ (the void configuration) i.o..

Invariant measure on S

successive visits to $\emptyset \Longrightarrow$ i.i.d. structure

$$R := \inf\{t > 0 : \varphi_t = \emptyset, \varphi_{t-} \neq \emptyset\}.$$

For $C \in \mathcal{B}(S)$:

$$m(C) := E_\emptyset \left[\int_0^R 1_C(\varphi_s) ds \right].$$

occupation time of C by the process during one life cycle

“Counting Invariant measure” on $I\!R^d$

For $B \in \mathcal{B}(I\!R^d)$:

$$\bar{m}(B) := E_{\emptyset}\left[\int_0^R \varphi_s(B)ds\right],$$

where

$$\varphi_s(B) := \sum_{k=1}^{l_s} 1_B(\varphi_s^k) : \text{ number of particles in } B \text{ at time } s$$

$\bar{m}(B)$ measures <u>total occupation time in B</u> of all particles during one life cycle of the process

Goal

- study of m and of $\bar{m}$
- in particular : existence of continuous (smooth?) Lebesgue density of m or $\bar{m}$

Plan of the talk

1. Statistics on branching diffusions

   Control of m and $\bar{m}$ important in parametric and non-parametric situations

2. Study of $\bar{m}$

   - "non-interactive" case: connection with resolvant, Hörmander's theorem
   - "interactive" case: Malliavin calculus

3. Study of m

   report on ongoing work with R. Höpfner

I. Statistics of BDI processes

I.1. Parametric case (Lö 2002, Ann I.H.P.)

$$\vartheta \in \Theta \subset I\!\!R^k \text{ open }, b = b^\vartheta, \kappa = \kappa^\vartheta, \text{ etc }.$$

Suppose

- regularity conditions,
  e.g. logarithmic differentiability of $\xi \mapsto \kappa^\xi(.)$ in an $L^2(\bar{m})-$sense, at $\xi = \vartheta$, with derivative $\dot{\kappa}^\vartheta$.
- integrability of information
  with respect to m or $\bar{m}$: e.g.

$$I^\vartheta(x) := \dot{\kappa}^\vartheta(\dot{\kappa}^\vartheta)^T \kappa^\vartheta(x), \; I^\vartheta \in L^1(\bar{m}).$$

analogous assumptions on the other ingredients.

Then: (when observing as $t \to \infty$) :

$$\text{LAN } (\vartheta), \text{ at rate } \frac{1}{\sqrt{t}}.$$

Hence : One has to control integrability w.r.t. m or $\bar{m}$.

I.2. Non-parametric case (Höpfner, Hoffmann, Lö 2002, S.J.S.)

dimension $d = 1$!

Goal : Non-parametric estimation of unknown branching rate $\kappa(.)$

Nadaraya-Watson kernel estimator:

$$K_{t,h}(a) := \frac{\int_0^t \int_{I\!R} \frac{1}{h} Q(\frac{y-a}{h}) \mu^{br}(dt, dy)}{A_t},$$

where

h bandwidth, μ^{br} branching point measure,

$A_t \sim$ total occupation time of neighbourhood of a of radius h

during time $[0, t]$

Suppose: $\kappa(.) \in H(\beta)$: Hölder class of known smoothness of order β

Theorem (Hö, Ho, Lö 2002)

1) If $h = o(t^{-1/2\beta+1})$:

$$\sqrt{th}\,(K_{t,h}(a) - \kappa(a)) \to \mathcal{N}(0, \Lambda).$$

2) The rate $t^{-\beta/2\beta+1}$ is asymptotically optimal for the Minimax risk in an integrated L^2-error.

Main probabilistic tools for the proof:

- Uniform Ratio limit theorem

  (by local time techniques : $d = 1$!)

- Continuity of Lebesgue density of $\bar{m}$ in a

  (still using local time techniques)

What happens if $d > 1$?

II. Study of $\bar{m}$ – "counting" invariant measure

Non-interactive case: All particles evolve independently.

Balance equation :

$$(A^* - \alpha)\bar{m} = -cg \tag{1}$$

A^* : formal adjoint of A :

generator of diffusion $dX_t = b(X_t)dt + \sigma(X_t)dW_t$,

$\alpha(x) := \kappa(x)(1 - \sum kp_k(x))$ effective killing rate

g immigration density, c immigration rate .

Remarks:

- balance equation only true under basic independence assumptions
- no more "branching" particle system in balance equation

- (1) implies : $\bar{m}$ **IS** the resolvant of the diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

killed at rate α

- (1) has been completely studied by analysts:

Theorem of Hörmander gives condition for existence of **smooth Lebesgue density** of $\bar{m}$: general Hörmander condition!

Interactive case: Coexisting particles may interact

- in spatial motion
- in branching mechanism

Balance equation (1) no more true.

Hence: Use probabilistic methods: **Malliavin calculus** $\Longrightarrow$

$$\bar{m}(dx) = p(x)dx, \quad p \in C_b^0 \text{ (Lö (2002))}$$

under

- uniform ellipticity
- conditions on interactions: e.g. mean field type interactions

III. Study of m – ongoing work with Reinhard Höpfner

m is measure on

$$S = \bigcup_{l=0}^{\infty} (\mathbb{R}^d)^l : \text{ direct sum}$$

m admits smooth Lebesgue density means:

$$\forall l > 0 : \ m|(\mathbb{R}^d)^l(dx) = p_l(x)dx, \ p_l \text{ smooth on } (\mathbb{R}^d)^l.$$

Notation

- T_n successive jumps of the process
- $K(x, dy) := \mathcal{L}(\varphi_{T_n} | \varphi_{T_n-} = x)(dy), \ x, y \in S$ "jump kernel"
- for $x \in (\mathbb{R}^d)^l \subset S : x = (x^1, \ldots, x^l)$:

$$k(x) := (\sum_{i=1}^{l} \kappa(x^i)) + c,$$

$$R^k(x, dy) := E_x \left(\int_0^\infty e^{-\int_0^t k(\xi_s) ds} \delta_{\xi_t}(dy) \right)$$

resolvant of $l-$dimensional diffusion

-

$$(R^k k)(x, dy) := R^k(x, dy)k(y) = \mathcal{L}(\varphi_{T_{n+1}-} | \varphi_{T_n} = x)(dy)$$

Then:

$$m = \sum_{n \geq 0} \delta_{\emptyset} \left[K R^k k \right]^n R^k$$

where

- Σ_n : sum over generations
- $\delta_{\emptyset} \left[K R^k k \right]^n$: law of φ_{T_n}
- Thus: $\delta_{\emptyset} \left[K R^k k \right]^n R^k$: occupation time during $[[T_n, T_{n+1}[[$

Density of m???

Problem : branching events $\Longrightarrow$ at time T_n two particles may be at one single location

Thus: Singularity at time $T_n \Longrightarrow \mathcal{L}(\varphi_{T_n})$ **IS NOT** absolute continuous!

But : $l-$particle flow is **regularising** (under non-degeneracy hypotheses)

$\Longrightarrow$ immediately after T_n : there exists a density!

Initial singularity $\Longrightarrow$ density blows up on hyper plane:

$$D := \{x = (x^1, \ldots, x^l) \in S, l \geq 0, x^i \neq x^j \forall i \neq j\}$$

Theorem:

m **admits continuous Lebesgue density on** D,
But: density will in general blow up on D^c (Hö-Lö 2002)

ongoing work: show that density is C^∞ on D

Outlook

Do the results on m and $\bar{m}$ help us to understand better the

Palm kernel:

$x \in \mathbb{R}^d$, $z \in S$:

$P(x, dz) :=$ " conditional law of configuration under m, given that one particle is located at x"