ESTIMATION OF SYSTEM RELIABILITY FROM ACCELERATED EXPERIMENTS

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Abstract

The accelerated life models models for the multivariate failure data are considered. Methods of estimation of system reliability under the normal conditions using results of accelerated experiments are proposed, when models are considered as semiparametric.

1. Introduction

Suppose that time to failure of some system with statistically dependent components is long in the normal conditions of fonctionning and the estimator of the reliability function or some another reliability characteristic is needed. In this case some experiments under the higher stress conditions can be used to obtain a sufficiently large data. The estimation of reliability characteristics under the normal stress using accelerated experiments can be done if the model, describing dependence of the survival distribution on the stress, is constructed.

2. Accelerated life models for systems of two components

Suppose that a random vector $T(x(\cdot)) = (T_1(x(\cdot)), T_2(x(\cdot)))$ is a vector of failure times of components of some system with two statistically dependent components under the stress $x : [0, \infty) \to R^k$. Denote

$$F_{x(\cdot)}(t_1, t_2) = \mathbf{P}\left\{T_1(x(\cdot)) \geq t_1, T_2(x(\cdot)) \geq t_2\right\}$$

the reliability function of $T(x(\cdot)), F_{i,x(\cdot)}$ - the marginal reliability functions of $F_{x(\cdot)}$ $(i = 1, 2)$.

Suppose that G is some bivariate reliability function, G_i - the marginal reliability functions of G, $H_i = G_i^{-1}$ - the inverse functions of G_i $(i = 1, 2)$.

Definition 1. We'll say that on a set of stresses $\mathcal{E}$ the *first transformation model* is true if there exist functionals $r_i : \mathcal{E} \to (0, +\infty)$ and a bivariate survival function G such that for all $x(\cdot) \in \mathcal{E}$

$$F_{x(\cdot)}(t_1, t_2) = G\left\{\int_0^{t_1} r_1[x(\tau)]\, d\tau, \int_0^{t_2} r_2[x(\tau)]\, d\tau\right\}. \tag{1}$$

Definition 2. We'll say that on a set of stresses $\mathcal{E}$ *the second transformation model* is true, if there exist functionals $r_i : \mathcal{E} \to (0, \infty)$, a bivariate reliability function G and univariate "baseline" reliability functions F_{i0} such that for all $x(\cdot) \in \mathcal{E}$

$$F_{x(\cdot)}(t_1, t_2) = G\left\{\int_0^{t_1} r_1[x(\tau)]\, dH_1(F_{10}(\tau)), \int_0^{t_2} r_2[x(\tau)]\, dH_2(F_{20}(\tau))\right\}. \tag{2}$$

Definition 3. We'll say that on a set of stresses $\mathcal{E}$ the *third transformation model* is true, if there exist functionals $a_i : \mathcal{E} \to (0, \infty)$, a bivariate reliability function G and univariate "baseline" reliability functions F_{i0} such that for all $x(\cdot) \in \mathcal{E}$

$$F_{x(\cdot)}(t_1, t_2) = G\left\{H_1(F_{10}(t_1)) + \int_0^{t_1} a_1(x(\tau))\, d\tau, H_2(F_{20}(t_2)) + \int_0^{t_2} a_2(x(\tau))\, d\tau\right\}. \tag{3}$$

If $E_0 \subset \mathcal{E}$ is some set of constant in time stresses and the models (1), (2) or (3) are true on E_0, then for all $x \in E_0$ the following equalities are true respectively:

$$F_x(t_1, t_2) = G\left\{r_1(x)t_1,\ r_2(x)t_2\right\}, \tag{4}$$

$$F_x(t_1, t_2) = G\left\{r_1(x)H_1(F_{10}(t_1)), r_2(x)H_2(F_{20}(t_2))\right\}, \tag{5}$$

$$F_x(t_1, t_2) = G\left\{H_1(F_{10}(t_1)) + a_1(x)t_1, H_2(F_{20}(t_2)) + a_2(x)t_2\right\}. \tag{6}$$

If G is specified with known G_i and unknown association parameter α, for example,

$$G(t_1, t_2) = G_1(t_1)G_2(t_2)[1 + \alpha(1 - G_1(t_1))(1 - G_2(t_2))], \quad (-1 \le \alpha \le 1) \tag{7}$$

or

$$G(t_1, t_2) = \begin{cases} exp\left\{-\frac{1}{\alpha} log\left[(G_1(t_1))^{-\alpha} + (G_2(t_2))^{-\alpha} - 1\right]\right\}, & \text{if } \alpha > 0 \\ G_1(t_1)G_2(t_2), & \text{if } \alpha = 0, \end{cases} \tag{8}$$

generalizations of proportional hazards, additive hazards, logistic regression, generalized probit, **accelerated failure time** models can be obtained.

If G has, for example, the form (7), then the equalities (1), (2) and (3) give the models

$$F_{x(\cdot)}(t_1, t_2; \alpha) = \prod_{i=1}^{2} G_i\left\{\int_0^{t_i} r_i[x(\tau)]\, d\tau\right\}\left[1 + \alpha\prod_{i=1}^{2}\left(1 - G_i\left\{\int_0^{t_i} r_i[x(\tau)]\, d\tau\right\}\right)\right], \tag{9}$$

$$\begin{aligned} F_{x(\cdot)}(t_1, t_2; \alpha) = \prod_{i=1}^{2} G_i\left\{\int_0^{t_i} r_i[x(\tau)]\, dH_i(F_{i0}(\tau))\right\} \times \\ \times\left[1 + \alpha\prod_{i=1}^{2}\left(1 - G_i\left\{\int_0^{t_i} r_i[x(\tau)]\, dH_i(F_{i0}(\tau))\right\}\right)\right] \end{aligned} \tag{10}$$

and

$$\begin{aligned} F_{x(\cdot)}(t_1, t_2; \alpha) = \prod_{i=1}^{2} G_i\left\{H_i(F_{i0}(t_i)) + \int_0^{t_i} a_i(x(\tau))\, d\tau\right\} \times \\ \times\left[1 + \alpha\prod_{i=1}^{2}\left(1 - G_i\left\{H_i(F_{i0}(t_i)) + \int_0^{t_i} a_i(x(\tau))\, d\tau\right\}\right)\right]. \end{aligned} \tag{11}$$

In the same manner the function G of the form (8) can be used for model construction. From (2) and (3) we obtain the models :

$$F_{x(\cdot)}(t_1, t_2; \alpha) = \begin{cases} exp\left\{-\frac{1}{\alpha} log\left[\sum_{i=1}^{2} G_i^{-\alpha}\left(\int_0^{t_i} r_i[x(\tau)] dH_i(F_{i0}(\tau))\right) - 1\right]\right\}, \\ \quad \text{if } \alpha > 0, \\ \prod_{i=1}^{2} G_i\left(\int_0^{t_i} r_i[x(\tau)] dH_i(F_{i0}(\tau))\right), \\ \quad \text{if } \alpha = 0, \end{cases} \tag{12}$$

$$F_{x(\cdot)}(t_1, t_2; \alpha) = \begin{cases} exp\left\{-\frac{1}{\alpha} log\left[\sum_{i=1}^{2} G_i^{-\alpha}\left(H_i(F_{i0}(t_i)) + \int_0^{t_i} a_i[x(\tau)] d\tau\right) - 1\right]\right\}, \\ \quad \text{if } \alpha > 0, \\ \prod_{i=1}^{2} G_i\left(H_i(F_{i0}(t_i)) + \int_0^{t_i} a_i(x(\tau)) d\tau\right), \\ \quad \text{if } \alpha = 0. \end{cases} \tag{13}$$

Consider, for example, generalizations of the proportional hazards model. If each of components satisfy **the proportional hazard model**, i.e. $G_i(t_i) = e^{-t_i}$, $\quad H_i(p_i) = -log\, p_i$, then the equality (10) implies the model

$$F_{x(\cdot)}(t_1,t_2) = exp\left\{\int\limits_0^{t_1} r_1[x_1(\tau)]dlog\, F_{10}(\tau)\right\} exp\left\{\int\limits_0^{t_2} r_2[x_2(\tau)]dlog\, F_{20}(\tau)\right\} \times$$

$$\times\left[1+\alpha\left(1-exp\left\{\int\limits_0^{t_1} r_1[x_1(\tau)]dlog\, F_{10}(\tau)\right\}\right)\left(1-exp\left\{\int\limits_0^{t_2} r_2[x_2(\tau)]dlog\, F_{20}(\tau)\right\}\right)\right].$$

If $x(\tau)\equiv x$, then

$$F_x(t_1,t_2;\alpha) = (F_{10}(t_1))^{r_1(x)}\,(F_{20}(t_2))^{r_2(x)}\,[1+\alpha(1-(F_{10}(t_1))^{r_1(x)})(1-(F_{20}(t_2))^{r_2(x)})].$$

The marginal survival functions are

$$F_{1x(\cdot)}(t_1) = F_{x(\cdot)}(t_1,0) = (F_{10}(t_1))^{r_1(x)}, \quad F_{2x(\cdot)}(t_2) = F_{x(\cdot)}(0,t_2) = (F_{20}(t_2))^{r_2(x)}.$$

Our method of model construction gives the possibility to obtain the generalization of the proportional hazards model, given by Clayton & Cuzick (1985). Really, the equality (12) implies

$$F_{x(\cdot)}(t_1,t_2;\alpha) = \begin{cases} exp\left\{-\frac{1}{\alpha}log\left[\sum\limits_{i=1}^{2} exp\left(-\alpha\int_0^{t_i} r_i(x(\tau))dlog\, F_{i0}(\tau)\right)-1\right]\right\}, \\ \quad \text{if } \alpha>0, \\ \prod\limits_{i=1}^{2} exp\left\{\int\limits_0^{t_i} r_i(x(\tau))dlog\ (F_{i0}(\tau))\right\}, \\ \quad \text{if } \alpha=0. \end{cases} \tag{14}$$

If $x(t)\equiv x$, then this model can be written as follows:

$$F_x(t_1,t_2;\alpha) = \begin{cases} exp\left\{-\frac{1}{\alpha}log\left[(F_{10}(t_1))^{-\alpha r_1(x)}+(F_{20}(t_2))^{-\alpha r_2(x)}-1\right]\right\}, & \text{if } \alpha>0, \\ (F_{10}(t_1))^{r_1(x)}(F_{20}(t_2))^{r_2(x)}, & \text{if } \alpha=0. \end{cases}$$

This is the **model of Clayton-Cuzick**.

Consider generalizations of the logistic regression model. Taking $G_i(t_i) = (1+t_i)^{-1}$, $H_i(p_i) = {p_i}^{-1}-1$ in (12) we obtain the following model

$$F_{x(\cdot)}(t_1,t_2;\alpha) = \begin{cases} exp\left\{-\frac{1}{\alpha}log\,[\sum\limits_{i=1}^{2}\left(1+\int\limits_0^{t_i} r_i[x(\tau)]dF_{i0}^{-1}(\tau))\right)^{\alpha}-1]\right\}, \\ \quad \text{if } \alpha>0, \\ \prod\limits_{i=1}^{2}\left(1+\int\limits_0^{t_i} r_i[x(\tau)]dF_{i0}^{-1}(\tau))\right)^{-1}, \\ \quad \text{if } \alpha=0. \end{cases} \tag{15}$$

3. Accelerated Life Models for systems of m components

Suppose that a system consists of m components and $T(x(\cdot)) = (T_1(x(\cdot)),\ldots,T_m(x(\cdot)))$ is a vector of failure times of components under the stress $x(\cdot)$, $F_{x(\cdot)}$ - the reliability function of $T(x(\cdot))$, $F_{ix(\cdot)}$ -

the marginal reliability functions of $F_{x(\cdot)}$, $(i=1,2,\dots,m)$. The models described in the case $m=2$ can be generalised to the case of arbitrary m.

Consider the multivariate survival function of the form (see Johnson (1975), Voinov and Nikulin (1996)):

$$G(t_1,\dots,t_m)=\prod_{j=1}^{m}G_j(t_j)\Bigg[1+\sum_{j_1<j2}^{m-1}\sum^{m}\alpha_{j_1j_2}\{1-G_{j_1}(t_{j_1})\}\{1-G_{j_2}(t_{j_2})\}-$$

$$-\sum_{j_1<j_2<j_3}^{m-2}\sum^{m-1}\sum^{m}\alpha_{j_1j_2j_3}\{1-G_{j_1}(t_{j_1})\}\{1-G_{j_2}(t_{j_2})\}\times$$

$$\times\{1-G_{j_3}(t_{j_3})\}+\dots+(-1)^m\alpha_{12\dots m}\prod_{j=1}^{m}\{1-G_j(t_j)\}\Bigg],\quad t_i\geq 0, \tag{16}$$

where G_j are some parameter free survival functions. The constants α's satisfy the conditions:

$$1+\sum_{j_1<j_2}^{m-1}\sum^{m}c_{j_1}c_{j_2}\alpha_{j_1j_2}+\sum_{j_1<j_2<j_3}^{m-2}\sum^{m-1}\sum^{m}c_{j_1}c_{j_2}c_{j_3}\alpha_{j_1j_2j_3}+\dots+\left(\prod_{j=1}^{m}c_j\right)\alpha_{123\dots m}\geq 0,\quad c_j=\pm 1.$$

In the case $m=2$

$$G(t_1,t_2;\alpha)=G_1(t_1)G_2(t_2)[1+\alpha\{1-G_1(t_1)\}\{1-G_2(t_2)\}],$$

where $-1\leq\alpha\leq 1$. So the survival function (16) is the generalization of (7).

The multivariate distribution is completly specified by the marginal distribution functions $G_1,\dots,G_m$ and the association parameters α's. In the case of a two-dimensional distribution there is only one association parameter α.

Suppose that the failure time vector $T(x(\cdot))=(T_1(x(\cdot)),\dots,T_m(x(\cdot)))^{\mathrm{T}}$ depends on the covariates $x(\cdot)=(x_1(\cdot),\dots,x_m(\cdot))^{\mathrm{T}}$.

Denote

$$F_{x(\cdot)}(t_1,\dots,t_m)=\mathbf{P}\{T_1(x(\cdot))\geq t_1,\dots,T_m(x(\cdot))\geq t_m\}$$

the survival function of $T(x(\cdot))$. Generalizing the equalities (10) - (11) we obtain the following models for the **multivariate failure time** data:

$$F_{x(\cdot)}(t_1,\dots,t_m)=G\left\{\int_0^{t_1}r_1[x(\tau_1)]dH_1(F_{10}(\tau_1)),\dots,\int_0^{t_m}r_m[x(\tau_m)]dH_m(F_{m0}(\tau_m))\right\}, \tag{17}$$

$$F_{x(\cdot)}(t_1,\dots,t_m)=$$

$$=G\left\{H_1(F_{10}(t_1))+\int_0^{t_1}a_1(x(\tau_1))d\tau_1,\dots,H_m(F_{m0}(t_m))+\int_0^{t_m}a_m(x(\tau_m))d\tau_m\right\}, \tag{18}$$

where F_{i_0} are some unknown baseline survival functions, $H_i=G_i^{-1}$, $(i=1,2,\dots,m)$.

4. Estimation. Bivariate case

Consider, for example, the model (10) with known G_i, i.e. we suppose, that each component satisfy proportional hazards, logistic regression or some other model. The parameter α determines dependence of times to failure of components. If $x(\tau)\equiv x=const$, the model (10) can be written as follows

$$F_{x(\cdot)}(t_1,t_2;\alpha)=\prod_{i=1}^{2}G_i\{r_i(x)H_i(F_{i0}(t_i))\}\left[1+\alpha\prod_{i=1}^{2}(1-G_i(r_i(x)H_i(F_{i0}(t_i)))\right]. \tag{19}$$

The natural parametrization of the functions $r_i(x)$ is $r_i(x) = \exp(\beta_i^{\mathrm{T}} z)$, where $\beta_i = (\beta_{i1}, \ldots, \beta_{ik})^{\mathrm{T}}$ is a vector of unknown parameters, $z = (z_1, \ldots, z_k)^{\mathrm{T}}$ is a vector of stresses (i.e. $z_i = x_i$) or some functions of stresses: $z_j = \varphi_j(x_j)$ ($j = 1, \ldots, k$; $i = 1, 2$).

Suppose that an estimator of the reliability function $F_{x_0}(t_1, t_2)$ under the "normal" stress x_0 is needed and data of experiments under accelerated with respect to x_0 stresses $x_1, \ldots, x_m$ are available. A failure of some component can be a censoring for another and *vice versa*. We suppose that each system also can be taken from experiment at some moment. In this case both components are censored at the same moment.

Suppose that n_i systems are tested under the stress x_i. The data has the form $(X_{\nu ij}, \delta_{\nu ij})$, where $X_{\nu ij}$ is the moment of a failure or a censoring of the ν component of the jth system, tested under the stress x_i, $\delta_{\nu ij} = 1$, if $X_{\nu ij}$ is a failure and $\delta_{\nu ij} = 0$ if $X_{\nu ij}$ is a censoring.

Denote $N_{\nu ij}(t)$ the number of observed failures of νth component of the item (i, j) in the interval $[0, t]$, $t \geq 0$, $Y_{\nu ij}(t) = 1$ if the νth component of the item (i, j) is "at risk" just prior to the moment t, i.e. it has not failed and has not been censored in the interval $[0, t)$ and $Y_{\nu ij}(t) = 0$ otherwise. Suppose that $\mathcal{F}_{\nu t}$, $t \geq 0$ is the filtration generated by $\{N_{\nu ij},\ i = 1, \ldots, k; j = 1, \ldots, n_i\}$ and

$$\lambda_{\nu ij}(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbf{P}\left\{N_{\nu ij}(t+\varepsilon) - N_{\nu ij}(t) = 1 \mid \mathcal{F}_{\nu t}\right\}$$

is the intensity of the process $N_{\nu ij}$. Assume that the random intensity $\lambda_{\nu ij}$ satisfy the **multiplicative intensity** model

$$\lambda_{\nu ij}(t) = \alpha_{\nu i}(t) Y_{\nu ij}(t)$$

i.e. censoring or failure of the νth components of other systems doesn't change the conditional probability that the νth component of some system fails just after the moment t with condition that this system is not censored and not failed till the moment t. Here $\alpha_{\nu i}(t)$ is the failure rate of the νth component of systems, tested under the stress x_i.

Denote $\alpha_\nu(t) = -G'_\nu / G_\nu$, $F_{\nu i}$ - the reliability function of $T_\nu(x_i)$. The **Kaplan-Meier estimator** of the function $F_{\nu i}$ is the statistic

$$\hat{F}_{\nu i}(s) = \prod_{u \leq s} \left(1 - \frac{\Delta N_{\nu i}(u)}{Y_{\nu i}(u)}\right) = \prod_{l : T_{\nu l}^{(i)} \leq s} \left(1 - \frac{d_{\nu l}^{(i)}}{Y_{\nu l}^{(i)}}\right),$$

where

$$N_{\nu i}(u) = \sum_{j=1}^{n_i} N_{\nu ij}(u), \quad \Delta N_{\nu i}(u) = N_{\nu i}(u) - N_{\nu i}(u-0), \quad T_{\nu 1}^{(i)} < T_{\nu 2}^{(i)} < \ldots < T_{\nu M_i}^{(i)}$$

are the moments of distinct failures of the νth component in the ith experiment, $d_{\nu l}^{(i)}$ - the number of failures at the moment $T_{\nu l}^{(i)}$, $Y_{\nu l}^{(i)}$ - the number of νth components "at risk" in the ith experiment at the moment $T_{\nu l}^{(i)}$. Determine the estimator (still depending on β_i) of the function $H_\nu(F_{0\nu}(t))$:

$$H_\nu(\hat{F}_{\nu 0}(t)) = \int_0^t \frac{dN_\nu(s)}{\sum_{l=1}^k r_\nu(x_l)\alpha_\nu(H_\nu(\hat{F}_{\nu l}(s)))Y_{\nu l}(s)} = \sum_{i=1}^k \sum_{j : T_{\nu j}^{(i)} \leq t} \frac{d_{\nu j}^{(i)}}{\sum_{l=1}^k e^{\beta_\nu^{\mathrm{T}} x_l} \alpha_\nu(H_\nu(\hat{F}_{\nu l}(T_{\nu j}^{(i)}))) Y_{\nu l}^{(i)}(T_{\nu j}^{(i)})},$$

where

$$N_\nu(s) = \sum_{i=1}^k N_{\nu i}(s).$$

Denote $I(A)$ the indicator of the event A,

$$\varepsilon_{1ij} = I\{\delta_{1ij} = 1\} I\{\delta_{2ij} = 1\},\ \varepsilon_{2ij} = I\{\delta_{1ij} = 1\} I\{\delta_{2ij} = 0\},$$

$$\varepsilon_{3ij} = I\{\delta_{1ij} = 0\} I\{\delta_{2ij} = 1\},\ \varepsilon_{4ij} = I\{\delta_{1ij} = 0\} I\{\delta_{2ij} = 0\},$$

$$\hat F_i(t_1,t_2;\alpha,\beta)=\prod_{\nu=1}^{2} G_i\left\{e^{\beta_\nu^{\mathrm T} z_i} H_\nu(\hat F_{\nu 0}(t_\nu))\right\}\left[1+\alpha\prod_{\nu=1}^{2}\left\{1-G_\nu(e^{\beta_\nu^{\mathrm T} z_i} H_\nu(\hat F_{\nu 0}(t_\nu)))\right\}\right].$$

Then the likelihood function for estimation of the parameters β and unknown **association parameter** α can be written as follows

$$L(\beta,\alpha)=\prod_{i=1}^{k}\prod_{j=1}^{n_i}[\hat F_i(X_{1ij}+0,X_{2ij}+0;\alpha,\beta)+\hat F_i(X_{1ij},X_{2ij};\alpha,\beta)-$$

$$-\hat F_i(X_{1ij}+0,X_{2ij};\alpha,\beta)-\hat F_i(X_{1ij},X_{2ij}+0;\alpha,\beta)]^{\varepsilon_{1ij}}\times$$

$$\times[\hat F_i(X_{1ij},X_{2ij};\alpha,\beta)-\hat F_i(X_{1ij}+0,X_{2ij};\alpha,\beta)]^{\varepsilon_{2ij}}\times$$

$$\times[\hat F_i(X_{1ij},X_{2ij};\alpha,\beta)-\hat F_i(X_{1ij},X_{2ij}+0;\alpha,\beta)]^{\varepsilon_{3ij}}[\hat F_i(X_{1ij},X_{2ij};\alpha,\beta)]^{\varepsilon_{4ij}}. \tag{20}$$

Maximizing this function, maximum likelihood estimators $\hat\beta$ and $\hat\alpha$ can be obtained.
The estimator of the reliability function F_{x_0} is

$$\hat F_{x_0}(t_1,t_2)=\prod_{\nu=1}^{2} G_\nu\left\{e^{\hat\beta_\nu^{\mathrm T} z_0} H_\nu(\hat F_{\nu 0}(t_\nu)\right\}\left[1+\hat\alpha\prod_{\nu=1}^{2}\left\{1-G_\nu\left\{e^{\hat\beta_\nu^{\mathrm T} z_0} H_\nu(\hat F_{\nu 0}(t_\nu))\right\}\right\}\right].$$

If instead of the model (19) the model

$$F_{x(\cdot)}(t_1,t_2;\alpha)=\prod_{\nu=1}^{2} G_\nu\left\{H_\nu(F_{\nu 0}(t_\nu))+a_\nu(x)\,t_\nu\right\}\left[1+\alpha\prod_{\nu=1}^{2}(1-G_\nu(H_\nu(F_{\nu 0}(t_\nu))+a_\nu(x)\,t_{nu}))\right]$$

which is a special case of the model (11) for constant stresses, is considered, the estimation can be done in the similar manner as in the case of the model (19). The likelihood function is (20), where

$$\hat F_i(t_1,t_2;\alpha,\beta)=\prod_{i=1}^{2} G_\nu\left\{H_\nu(\hat F_{\nu 0}(t_\nu))+\beta_\nu^{\mathrm T} z_i\right\}\left[1+\alpha\prod_{i=1}^{2}\left(1-G_\nu\left\{H_\nu(\hat F_{\nu 0}(t_\nu))+\beta_\nu^{\mathrm T} z_i)\right\}\right)\right]$$

and

$$H_\nu(\hat F_{\nu 0}(t))=\sum_{i=1}^{k}\int_0^t\frac{dN_{\nu i}(s)-\alpha_\nu(H_\nu(\hat F_{\nu i}(s)))Y_{\nu i}(s)\beta_\nu^{\mathrm T} z_i ds}{\sum_{j=1}^{k}\alpha_\nu(H_\nu(\hat F_{\nu i}(s)))Y_{\nu ij}(s)}.$$

5. Estimation. Multivariate case

Consider, for example, the model (19). The unknown functions F_{j0} and the association parameters α must be estimated. Estimation of functions F_{j0} can be done without an experiment of the whole system. For example, if $K_{\nu i}$ units of the νth component are tested under the stress x_i, the data is of the form

$$(X_{\nu ij},\delta_{\nu ij}),\quad (\nu=1,2,\dots,m;\ i=1,\dots,k;\ j=1,\dots,K_{\nu i}).$$

In this case the coefficients $\beta_\nu=(\beta_{\nu 1},\dots,\beta_{\nu k})^{\mathrm T}$ can be estimated by solving the score equations

$$U_\nu(\beta_\nu)=\sum_{i=1}^{m}\int_0^\infty z_i\left\{dN_{\nu i}(t)-Y_{\nu i}(t)exp\left\{\beta_\nu^{\mathrm T} z_i\right\}\alpha_\nu\left\{exp\left\{\beta_\nu^{\mathrm T} z_i\right\}H_\nu(\hat F_{\nu 0}(t))\right\}dH_\nu(\hat F_{\nu 0}(t))\right\}=0,$$

which is equivalent to the system of equations

$$\sum_{i=1}^{m}\sum_{l=1}^{K_{\nu i}} z_i[d_{\nu il}-Y_{\nu i}(T_{\nu il})e^{\beta_\nu^{\mathrm T} z_i}\alpha_\nu(e^{\beta_\nu^{\mathrm T} z_i}H_\nu(\hat F_{\nu 0}(T_{\nu il}))(H_\nu(\hat F_{\nu 0}(T_{\nu il}+0))-H_\nu(\hat F_{\nu 0}(T_{\nu il})))]=0.$$

If there are many identical elements, then the number of experiments is considerably smaller. For example, if all the components are identical elements, then $H_\nu(F_{\nu 0}(t))$ are all equal and the experiment consists in testing of several groups of these elements under different stresses x_i. In this case

$$H_\nu(\hat{F}_{\nu 0}(t)) = H_1(\hat{F}_{10}(t)),\ \nu = 2, \ldots, m.$$

The estimation of the association parameters α can be done if there are some accelerated experiments of systems. Many of these parameters can be taken equal. This depends on the manner of assembling of these components into a system. Suppose that n_i systems are tested under the stress x_i $(i = 1, \ldots, k)$. As only the estimators of parameters α are needed, $k = 1$ can be often taken. The data is

$$(X_{\nu ij}, \delta_{\nu ij})\ (\nu = 1, \ldots, m;\ i = 1, \ldots, k;\ j = 1, \ldots, n_i).$$

Denote

$$\hat{F}_i(t_1, \ldots, t_m; \alpha) = \prod_{j=1}^{m} G_j \left\{ e^{\hat{\beta}_j^{\mathrm{T}} z_i} H_j(\hat{F}_{j0}(t_j)) \right\} \left[1 + \sum_{j_1<j_2}^{m-1} \sum^{m} \alpha_{j_1 j_2} \prod_{l=1}^{2} \left\{ 1 - G_{j_l}(e^{\hat{\beta}_{j_l}^{\mathrm{T}} z_i} H_{j_l}(t_{j_l}))) \right\} + \ldots + \right.$$

$$\left. + (-1)^m \alpha_{12\ldots m} \prod_{j=1}^{m} \left(1 - G_j \left\{ e^{\hat{\beta}_j^{\mathrm{T}} z_i} H_j(\hat{F}_{j0}(t_j)) \right\} \right) \right],$$

$$Z^{ij}(\alpha) = F_i(X_{1ij}, \ldots, X_{mij}; \alpha),\ Z_r^{ij}(\alpha) = F_i(X_{1ij}, \ldots, X_{r-1,ij}, X_{rij} + 0, X_{r+1,ij}, \ldots, X_{mij}; \alpha),$$

$$Z_{r_1 r_2}^{ij}(\alpha) = F_i(X_{1ij}, \ldots, X_{r_1-1,ij}, X_{r_1 ij} + 0, X_{r_1+1,ij}, \ldots, X_{r_2-1,ij}, X_{r_2 ij} + 0, X_{r_2+1,ij}, \ldots, X_{mij}; \alpha),$$

etc. Then the likelihood function for the estimation of α is

$$L(\alpha) = \prod_{i=1}^{k} \prod_{j=1}^{n_i} \left\{ Z^{ij}(\alpha) - \sum_{r=1}^{m} \delta_{rij} Z_r^{ij}(\alpha) + \sum_{r_1<r_2}^{m-1} \sum^{m} \delta_{r_1 ij} \delta_{r_2 ij} Z_{r_1 r_2}^{ij}(\alpha) - \right.$$

$$\left. - \sum_{r_1<r_2<r_3}^{m-2} \sum^{m-1} \sum^{m} \delta_{r_1 ij} \delta_{r_2 ij} \delta_{r_3 ij} Z_{r_1 r_2 r_3}^{ij}(\alpha) + \ldots + (-1)^m \delta_{1ij} \delta_{2ij} \ldots \delta_{mij} Z_{12\ldots m}^{ij}(\alpha) \right\}.$$

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