Nonparametric estimation for diffusions from low-frequency data is ill-posed

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Overview:

1. Scalar diffusions

2. High-frequency estimation

3. Different observation schemes

4. Low-frequency estimation

5. Simulations

6. Outlook

1 Scalar diffusions

Itô: Solution of stochastic differential equation

$$\boxed{X_t = X_0 + \int_0^t b(X_u, u)\,du + \int_0^t \sigma(X_u, u)\,dW_u.}$$

Feller: continuous Markov process with infinitesimal conditional moments ($\Delta \to 0$)

$$\begin{aligned}\mathbb{E}[X_{t+\Delta} - X_t \,|\, X_t] &= b(X_t, t)\Delta + o(\Delta),\\ \mathrm{Var}[X_{t+\Delta} - X_t \,|\, X_t] &= \sigma^2(X_t, t)\Delta + o(\Delta).\end{aligned}$$

Assumption.
$(X_t,\, t \geq 0)$ is time-homogeneous: $b(x,t) = b(x)$, $\sigma(x,t) = \sigma(x)$.

2 High-frequency estimation

Observation:

$(X_{t_n}, n \le N)$ with $t_n = n\Delta$; $N \to \infty$, $\Delta \to 0$

Empirical drift:

$$\frac{1}{\Delta}\Big(X_{t_{n+1}} - X_{t_n}\Big) = \frac{1}{\Delta}\Big(\int_{t_n}^{t_{n+1}} b(X_u)\,du + \int_{t_n}^{t_{n+1}} \sigma(X_u)\,dW(u)\Big)$$
$$\approx b(X_{t_n}) + \sigma(X_{t_n})\Delta^{-1/2}\varepsilon_n, \quad \varepsilon_n \overset{iid}{\sim} N(0,1).$$

$\longrightarrow$ nonlinear autoregression model

Estimators $\hat{b}$ and $\hat{\sigma}$: obtained by smoothing $(\widehat{b(X_{t_n})})_n$ and $(\widehat{\sigma(X_{t_n})})_n$, respectively.

3 Different observation schemes

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad t \geq 0$$

Asymptotic theory under mixing:

continuous time (CT) observation:

$(X_t, 0 \leq t \leq T)$, $T \to \infty$

high-frequency (HF) observation:

$(X_{n\Delta}, 0 \leq n \leq N)$, $\Delta \to 0$, $N\Delta \to \infty$

low-frequency (LF) observation:

$(X_{n\Delta}, 0 \leq n \leq N)$, $N \to \infty$ (Δ fixed)

Overview about minimax rates

(regular parametrisation vs. functions of smoothness s)

observation	Parametric		Nonparametric	
	b	σ	b	σ
CT	$T^{-\frac{1}{2}}$	known	$T^{-\frac{s}{2s+1}}$	known
HF	$(N\Delta)^{-\frac{1}{2}}$	$N^{-\frac{1}{2}}$	$(N\Delta)^{-\frac{s}{2s+1}}$	$N^{-\frac{s}{2s+1}}$
LF	$N^{-\frac{1}{2}}$	$N^{-\frac{1}{2}}$	???	???

4 Estimation from low-frequency data

Problem:

Only the time-discrete Markov process $(X_{n\Delta})_{n\geq 0}$ is observed.

$$\boxed{\text{observation } (X_{n\Delta})_{0\leq n\leq N}}$$

$$\Big\downarrow \text{estimation}$$

$$\boxed{\text{transition operator } P_\Delta f(x) = \mathbb{E}_{\sigma,b}[f(X_\Delta)\,|\,X_0 = x]}$$

$$\Big\downarrow \text{identification}$$

$$\boxed{\text{parameter } \sigma(\cdot),\ b(\cdot)}$$

4.1 Identification under reflection

Assumption. X is a diffusion on $[0,1]$ with reflection. The functions $b, \sigma : [0,1] \to \mathbb{R}$ are for $s > 1$, $R > c > 0$ in the class

$$\Theta_s := \left\{ (b,\sigma) \,\middle|\, \|b\|_{H^{s-1}} \le R,\ \|\sigma\|_{H^s} \le R,\ \sigma(\cdot) \ge c \right\}.$$

invariant density:

$$\mu(x) = \frac{2C}{\sigma^2(x)} \exp\left(\int_0^x \frac{2b(u)}{\sigma^2(u)}\, du \right), \qquad x \in [0,1]$$

scale density:

$$S(x) = C^{-1} \exp\left(-\int_0^x \frac{2b(u)}{\sigma^2(u)}\, du \right), \qquad x \in [0,1]$$

infinitesimal generator on $L^2(\mu)$:

$$Lf(x) = \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x) = \frac{(S^{-1}\cdot f')'(x)}{\mu(x)}$$

$$\mathcal{D}(L) = \{f \in H^2([0,1]) \,|\, f'(0) = f'(1) = 0\}.$$

identification via eigenfunction u_1 of P_Δ:

$$P_\Delta u_1 = \lambda_1 u_1 \Rightarrow Lu_1 = \nu_1 u_1 \quad (\nu_1 = \Delta^{-1}\log(\lambda_1))$$

$$\Rightarrow \boxed{\begin{aligned} S(x) &= \frac{u_1'(x)}{\nu_1 \int_0^x u_1(y)\mu(y)\,dy} \\ \sigma^2(x) &= \frac{2S^{-1}(x)}{\mu(x)}, \qquad b(x) = \frac{(S^{-1})'(x)}{\mu(x)} \end{aligned}}$$

(Hansen/Scheinkman/Touzi, 1998)

4.2 Plug-in estimator

$\widehat{\mu}$: classical density estimation

$(\widehat{\nu}_1, \widehat{u}_1)$: spectral estimation for P_Δ
Projection method on basis functions $e_1, \ldots, e_M$:

$$\widehat{P}_\Delta \in \mathbb{R}^{M\times M} \text{ with } \quad (\widehat{P}_\Delta)_{k,l} := \frac{1}{N}\sum_{n=1}^{N} e_k(X_{t_m})e_l(X_{t_{m-1}})$$

$$\xrightarrow{N\to\infty} \mathbb{E}[e_k(X_\Delta)e_l(X_0)] = \langle P_\Delta e_k, e_l\rangle_\mu$$

$$\widehat{G} \in \mathbb{R}^{M\times M} \text{ with } \quad \widehat{G}_{k,l} := \frac{1}{N}\sum_{n=1}^{N} e_k(X_{t_m})e_l(X_{t_m})$$

$$\xrightarrow{N\to\infty} \mathbb{E}[e_k(X_0)e_l(X_0)] = \langle e_k, e_l\rangle_\mu$$

$\to$ Find eigenpair $(\hat{\lambda}_1, \hat{u}_1)$ of $\widehat{G}^{-1}\widehat{P}_\Delta$ and put $\widehat{\nu}_1 = \Delta^{-1}\log(\hat{\lambda}_1)$.

4.3 Minimax rates

Analysis: $(\sigma, b) \in \Theta_s \Rightarrow \mu \in H^s,\ u_1 \in H^{s+1}$

$\widehat{\mu}$: asymptotic L^2-risk is $N^{-s/(2s+1)}$

$(\widehat{\nu}_1, \widehat{u}_1)$: asymptotic L^2-risk is $N^{-(s+1)/(2s+3)}$

BUT: need u_1' for σ^2 and u_1'' for $b \Rightarrow$ ill-posed

Theorem. *For* $0 < A < B < 1$ *the plug-in estimators attain the minimax rates over* Θ_s*:*

$$\inf_{\widehat{\sigma}} \sup_{(\sigma,b)\in\Theta_s} \mathbb{E}_{\sigma,b}\Big[\|\widehat{\sigma}-\sigma\|^2_{L^2([A,B])}\Big]^{1/2} \sim N^{-s/(2s+3)},$$

$$\inf_{\widehat{b}} \sup_{(\sigma,b)\in\Theta_s} \mathbb{E}_{\sigma,b}\Big[\|\widehat{b}-b\|^2_{L^2([A,B])}\Big]^{1/2} \sim N^{-(s-1)/(2s+3)}.$$

Remarks:

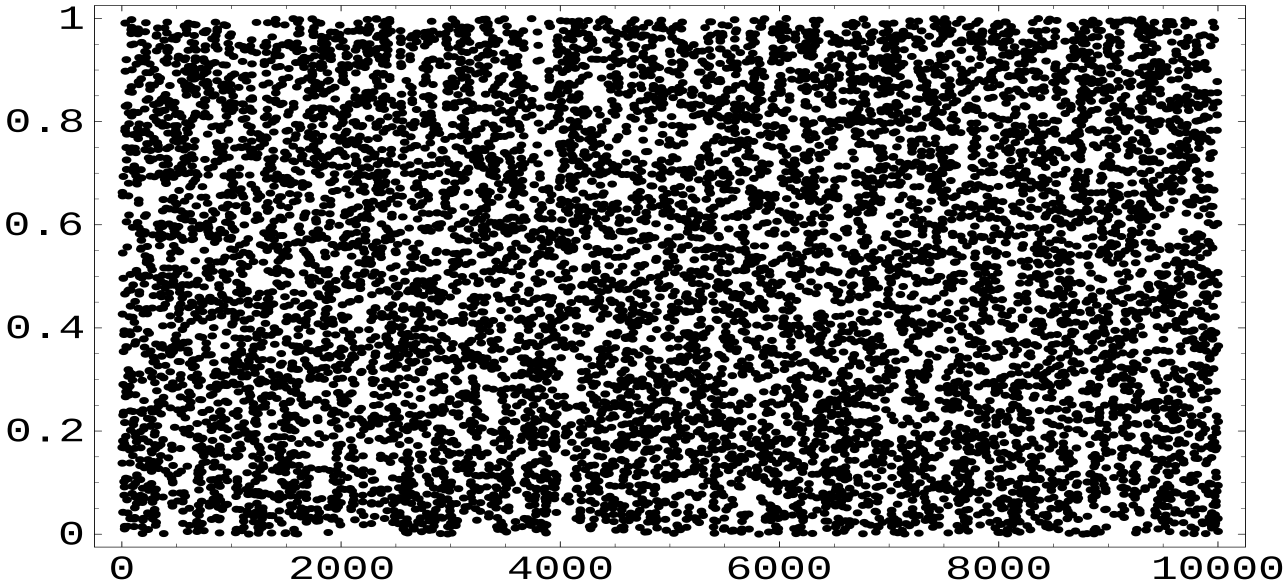
Estimation of σ is ill-posed of degree 1.

Estimation of b is ill-posed of degree 2.

Rate is worse if $A = 0$ or $B = 1$.

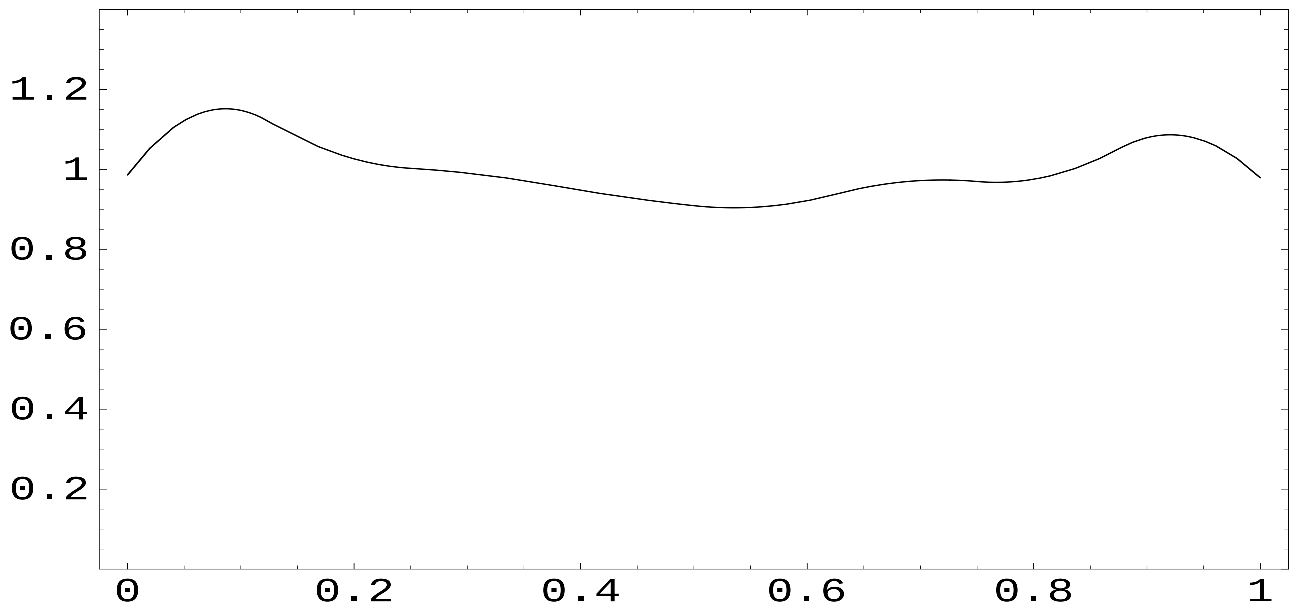
Rate for b depends on σ and vice versa.

Lower bound follows from resolvent estimates for local alternatives of Brownian motion with the same invariant measure.

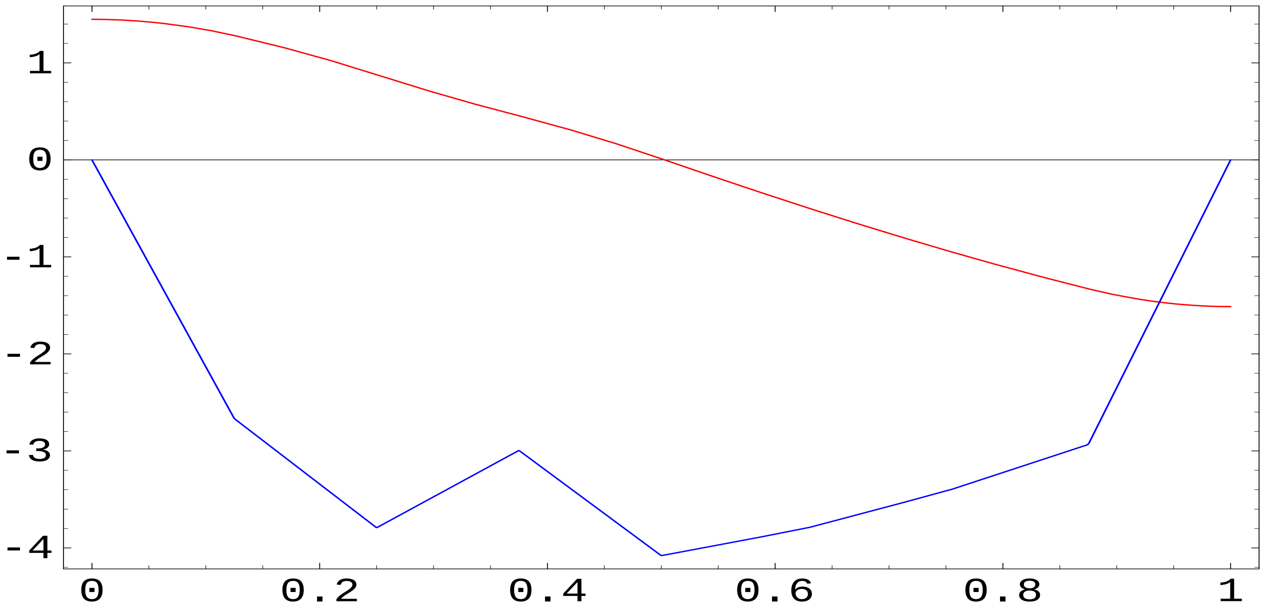


$$dX_t = (1 - 2X_t)dt + (\tfrac{1}{2} + 2x(1-x))dW_t$$

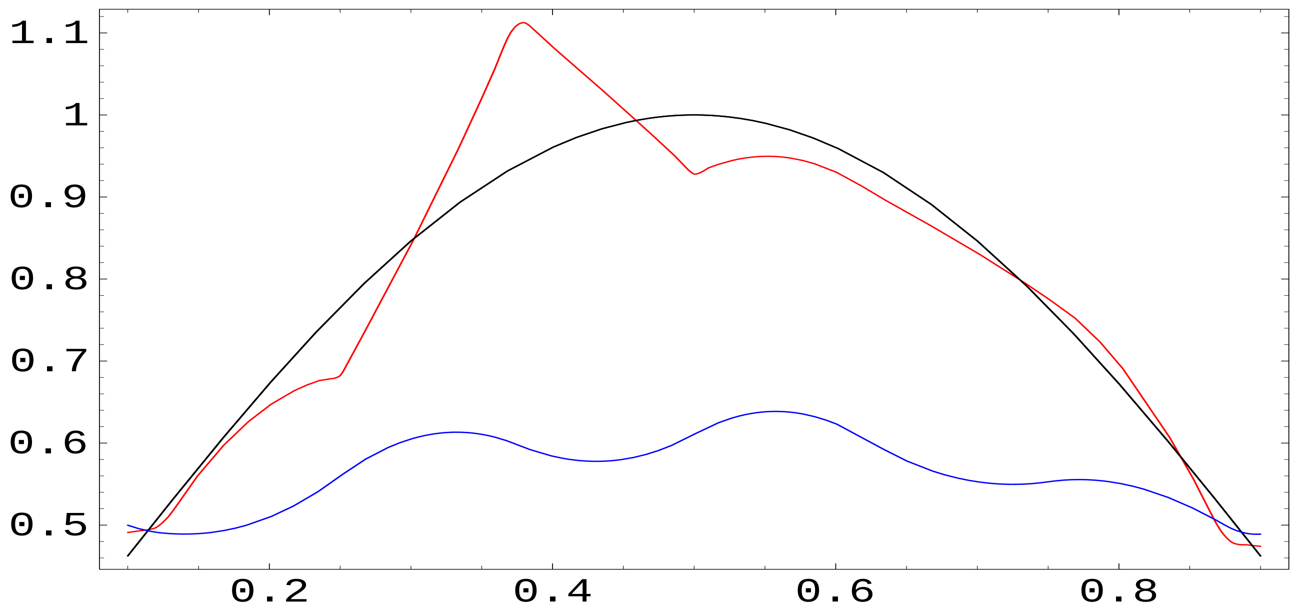
$$(\Delta = 0.1;\ N\Delta = 1000)$$



Estimation of the invariant density $\widehat{\mu}$
(using quadratic splines)

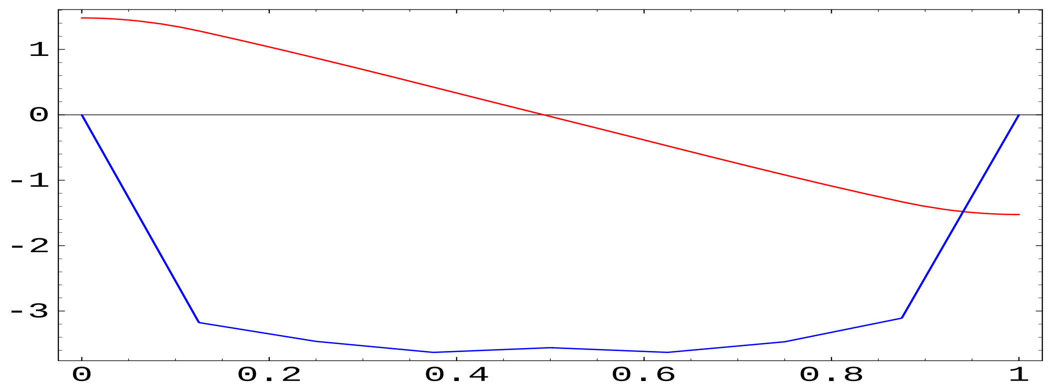


Estimation of the eigenfunction $\widehat{u}_1$ with derivative $\widehat{u}_1'$
(using quadratic splines, $\Delta = 0.1$)

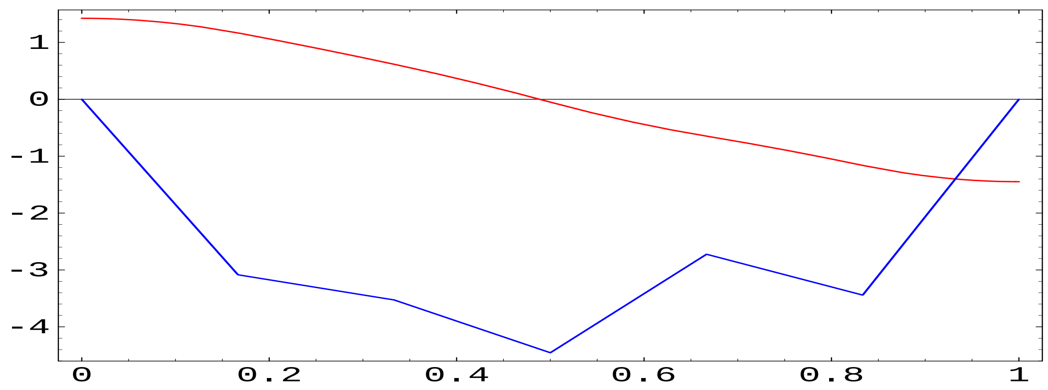


diffusion coefficient σ^2 with LF- und HF-estimator ($\Delta = 0.1$)

Eigenfunction $\widehat{u}_1$ and $\widehat{u}_1'$

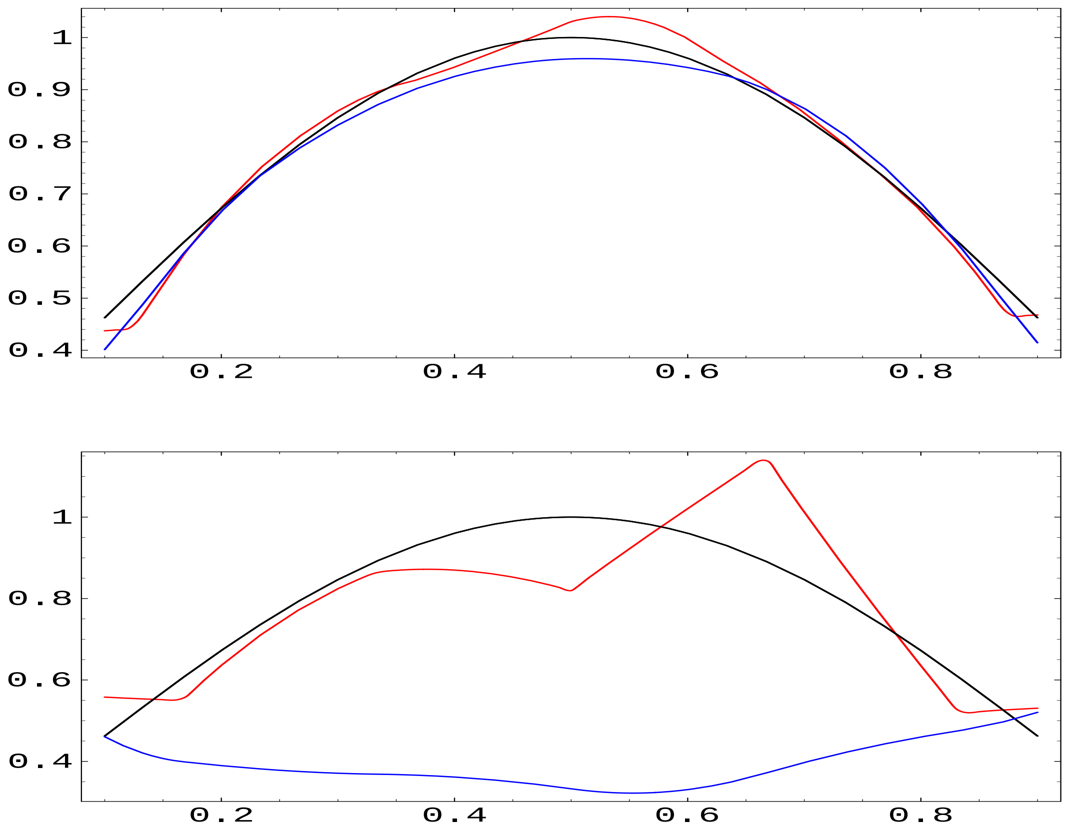


$\Delta = 0.01$



$\Delta = 0.25$

LF-/HF-estimation of σ^2



$\Delta = 0.01$

$\Delta = 0.25$

6 Outlook

We have closed the gap and obtain roughly:

obs.	Parametric		Nonparametric	
	b	σ	b	σ
CTS	$T^{-\frac{1}{2}}$	known	$T^{-\frac{s}{2s+1}}$	known
HF	$(N\Delta)^{-\frac{1}{2}}$	$N^{-\frac{1}{2}}$	$(N\Delta)^{-\frac{s}{2s+1}}$	$N^{-\frac{s}{2s+1}}$
LF	$N^{-\frac{1}{2}}$	$N^{-\frac{1}{2}}$	$N^{-\frac{s}{2s+5}}$	$N^{-\frac{s}{2s+3}}$

but...

adaptive estimation?

extension to $\mathbb{R}$ or even $\mathbb{R}^d$ instead of $[0,1]$?

application to real data?

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