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Estimation for discretely observed diffusion process with jumps

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— Joint work (Sections 1-4) with Yasutaka Shimizu

Consider a d-dimensional diffusion process $X.$ with jumps defined by the SDE:

$$
\boxed{
\begin{aligned}
dX_t &= a(X_t,\theta)\,dt + b(X_{t-},\sigma)\,dW_t \\
&\quad + \int_E c(X_{t-},z,\theta)\,(p-q^\theta)(dt,dz) \\
X_0 &= x_0,
\end{aligned}
}
$$

where
$E = \mathbf{R}^k \setminus \{0\}$,
$\alpha = (\theta,\sigma) \in \Xi \subset \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$ (parameter space)
$(W_t)_{t\geq 0}$ is a r-dimensional standard $\mathcal{F}_t$-Brownian motion ,
$p(dt,dz)$ is a time-homogeneous Poisson random measure on $\mathbf{R}_+ \times \mathbf{R}^k$,
$q^\theta(dt,dz)$ is its intensity measure.

We set

$$
q^\theta(dt,dz) = f_\theta(z)\,dz\,dt.
$$

One of the most simple but important examples of one dimensional case is wrriten as follows.

$$dX_t = \tilde{a}(X_t, \mu)\, dt + \tilde{b}(X_{t-}, \sigma)\, dW_t + \tilde{c}(X_{t-}, \vartheta)\, dZ_t^{\vartheta}, \quad (1)$$

where $Z_\cdot^{\vartheta}$ is a Lévy process with parameter ϑ. This belongs to the class of (1) with $c(x, z, \theta) = \tilde{c}(x, \vartheta)z$, $a(x, \theta) = \tilde{a}(x, \mu) + \int_E \tilde{c}(x, \vartheta) z f_\vartheta(z)\, dz$ and $\theta = (\mu, \vartheta)$.

In this paper, we studied estimation of the parameter $\alpha = (\theta, \sigma)$ from discrete observations. For this aim, we observe $n+1$ data $(X_{t_i^n})_{0 \le i \le n}$, and show the consistency and asymptotic normality of an estimator under $h_n \to 0,\ nh_n \to \infty,\ nh_n^2 \to 0.$

$t_i^n = ih_n.$

1 Notations and Assumptions

We will introduce some notations .

1. We set $\alpha = (\theta, \sigma)$, and $\alpha_0, \theta_0, \sigma_0$ denotes the respective true values.

2. For $\kappa = (\kappa_1, \ldots, \kappa_d)$,
$\partial_{\kappa_j} := \dfrac{\partial}{\partial \kappa_j},\ \partial^2_{\kappa_j} := \dfrac{\partial^2}{\partial \kappa_j^2},\ \partial^2_{\kappa_i \kappa_j} := \dfrac{\partial^2}{\partial \kappa_i \partial \kappa_j}$,
$\partial_\kappa := (\partial_{\kappa_1}, \ldots, \partial_{\kappa_d})^*$, and $\partial^2_\kappa f(\kappa) = \left(\partial^2_{\kappa_i \kappa_j} f(\kappa)\right)_{1 \leq i,j \leq d}$,
where $*$ stands for the transpose.

3. For g which is a function defined on $\mathbf{R}^d \times \Xi$, we denotes by $g_{i-1}(\alpha)$ the value $g(X_{t^n_{i-1}}, \alpha)$.

4. $\Delta X^n_i := X_{t^n_i} - X_{t^n_{i-1}}, \quad \Delta X_t := X_t - X_{t-}$.

5. $\beta(x, \sigma) := b(x, \sigma) b^*(x, \sigma)$.

6. For a tensor A, $|A|^2$ is the sum of squares of the components of A.

7. Let u_n be a real valued sequense.
$R : \Xi \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ denotes a function for which there exists a constant C such that

$R(\alpha, u_n, x) \leq u_n C (1 + |x|)^C \quad$ for all $\alpha \in \Xi,\ x \in \mathbf{R}^d,\ n \in \mathbf{N}$

We set $\tilde{R}(\alpha, u_n, x) = 1 - R(\alpha, u_n, x)$.

8. Let b_n be a real valued sequense such that
$b_n \to \infty,\ nh_n^2 b_n \to 0,\ \dfrac{b_n}{nh_n} \to 0.$ For example
$h_n = n^{-2/3},\ b_n = n^{1/4}$ etc. Moreover, we set $\varepsilon_n = b_n^{-1/10}$.

9. In the following discussion, if we write X, then it means the solution to (1) with $\alpha = \alpha_0$.

We make the following assumptions.

A 1 *(i)**(Lipschitz condition)** For constant L, and function $\zeta(z)$ which is of polynomial growth in z,*

$$|a(x,\theta_0) - a(y,\theta_0)| + |b(x,\sigma_0) - b(y,\sigma_0)| \le L|x-y|,$$

$$|c(x,z,\theta_0) - c(y,z,\theta_0)| \le \zeta(z)|x-y|,$$

*(ii)**(Linear growthness)***

$$|a(x,\theta_0)|+|b(y,\sigma_0)| \le L(1+|x|), \quad |c(x,z,\theta_0)| \le \zeta(z)(1+|x|)$$

A 2 *The diffusion process with jumps X is ergodic and stationary for $\alpha = \alpha_0$: if $\pi(dx)$ is its invariant probability measure, the initial distribution is π and*

$$\frac{1}{T}\int_0^T f(X_t)\,dt \xrightarrow{P} \int f(x)\,d\pi,$$

where f has π-integrability. Moreover we suppose that $\int |x|^p\,\pi(dx) < \infty$ for all $p \ge 0$.

A 3 *For each* θ *and* x, *the mapping* $z \mapsto y = c(x, z, \theta)$ *has an inverse* $z = c^{-1}(x, y, \theta)$ *which is differentiable with respect to* y, *and we set*

$$\boxed{\Psi_\theta(y, x) := f_\theta\left(c^{-1}(x, y, \theta)\right) J(x, y, \theta),}$$

where $J(x, y, \theta)$ *is an absolute value of Jacbian of* $c^{-1}(x, y, \theta)$.

A 4 $\inf_{x,\sigma} \det \beta(x, \sigma) > 0$, $\inf_{x,\theta} |c(x, z, \theta)| \geq c_0|z|$ $(c_0 > 0)$, $\inf_\theta \lambda(\theta) > 0$. *Moreover we assume* β *is a positive definite.*

A 5 *The function* $a(x, \theta), b(x, \sigma)$ *are respectively twice continuously differentiable with respect to* x *for all* θ, σ, *and their derivatives are of polynomial growth in* x *uniformly in* α *for each component. Moreover* a, b *are respectively twice continuously differentiable with respect to their parameters for fixed* x, *and their all derivatives are also of polynomial growth in* x *uniformly in* α *for each component.*

A 6 *For fixed* x, y, *the function* $\Psi_\theta(y, x)$ *is three times continuously differentiable with respect to* θ, *for fixed* θ, x, *it is differentiable with respect to* y, *and, for fixed* θ, y, *it is differentiable with respect to* x. *Moreover, we* *assume that, for $k = 0, 1, 2,\ l = 0, 1,$*

$$|\partial_\theta^{k+l}\Psi_\theta(y,x)| \le L_1(y)(1+|x|)^C, \tag{2}$$

$$|\partial_x\partial_\theta^k\Psi_\theta(y,x)| \le L_2(\theta,y), \tag{3}$$

where L_1, L_2 are bounded and $d\pi$-integrable functions. Furthermore,

$$|\partial_y\partial_\theta^k\Psi_\theta(y,x)| \le C(1+|y|)^C(1+|x|)^C, \tag{4}$$

$$\int \sup_\theta \left|\partial_\theta^{k+l}\log\Psi_\theta(y,x)\Psi_{\theta_0}(y,x)\right| dy \le C(1+|x|)^C. \tag{5}$$

A 7 *For arbitrary $p \ge 0, h > 0$,*

$$\sup_t E[|X_t|^p] < \infty.$$

A 8 *We suppose the identifiability as follows.*

$$\begin{aligned}
&\det\beta(x,\sigma) = \det\beta(x,\sigma_0) \Longrightarrow \sigma = \sigma_0,\\
&\begin{cases} a(x,\theta) = a(x,\theta_0) \\ \Psi_\theta(y,x) = \Psi_{\theta_0}(y,x) \end{cases} \Longrightarrow \theta = \theta_0.
\end{aligned}$$

A 9 *There exists some constants $r, M > 0,\ \gamma > 3$ such that $\sup_\theta f_\theta(z)\mathbf{1}_{\{|z|\le r\}} \le M|z|^\gamma$, and*

$$\sup_\theta \int |z|^p f_\theta(z)\,dz < \infty$$

for all $p \ge 0$.

A 10 *A seaquence of real valued functions* $\{\varphi_n(x,y)\}_{n\in\mathbf{N}}$ *satisfies following properties: For fixed* n, $0 \le \varphi_n \le 1$ *and* $\varphi_n \to 1$ $dy \times d\pi$*-a.s. There exists some* $M > 0$ *such that,*

$$\varphi_n(x,y) = \begin{cases} 1 & (x,y) \in \left\{\inf_{\theta\in\Theta} \Psi_\theta(y,x) > M\right\} \\ 0 & (x,y) \in \left\{\inf_{\theta\in\Theta} \Psi_\theta(y,x) \le \varepsilon_n\right\} \end{cases}$$

Moreover,

$$\sup_{x,y} |\partial_x \varphi_n|, \quad \sup_{x,y} |\partial_y \varphi_n| \le O(\varepsilon_n^{-1}), \tag{6}$$

and, on $\left\{\inf_{\theta\in\Theta} \Psi_\theta(y,x) \le \varepsilon_n\right\}$,

$$\partial_x \varphi_n = 0, \quad \partial_y \varphi_n = 0. \tag{7}$$

Let us give some examples of $\Psi_\theta(y,x)$ and check conditions in A6.

Example 1 *We consider the following one-dimensinal stochastic differential equation.*

$$dX_t = a(X_t)\,dt + b(X_{t-})\,dW_t + c(X_{t-},\theta)\,dZ_t^\theta, \qquad (8)$$

where $Z_\cdot^\theta$ *is a Lévy process with Lévy density* $f_\theta(z)$. *We assume, for some* $M,\ K > 0$,

$$c(x,\theta) = \frac{1}{c}, \quad f_\theta(z) = Mz^\alpha(K-z)^\beta \mathbf{1}_{\{0\le z\le K\}},$$

and $\theta = (c,\alpha,\beta)$, *where* $c < c_0,\ \alpha,\ \beta > 3$.

$$\Psi_\theta(y,x) = Mc(cy)^\alpha(K-cy)^\beta,$$

$$\log\Psi_\theta(y,x) = \log M + (\alpha+1)\log c + \alpha\log y + \beta\log(K-cy)$$

Noticing the parameter space Θ *is compact, it is easy to see from (2) to (4). Moreover all* $\sup_\theta\left|\partial_\theta^{k+l}\log\Psi_\theta(y,x)\Psi_\theta(y,x)\right|$ *are bounded.*

Example 2 *For SDE (8), we assume*

$$c(x,\theta) = \frac{1}{c}, \quad f_\theta(z)\mathbf{1}_{\{|z|\le M\}} = e^{-\frac{\gamma}{z}}\mathbf{1}_{\{|z|\le M\}}.$$

and $\theta = (c,\gamma)$, *where* $c < c_0,\ \gamma > 0$. *For simplicity, we suppose that supp*$(f_\theta) \subset \mathbf{R}_+$ *and* f_θ *is symmetric with respect to some axis. Under these condition, it is enough to check A6 neighborhood of the origin. Then, on the set* $\{|z| \le M\}$ *for some* $M > 0$,

$$\Psi_\theta(y,x) = e^{-\frac{\gamma}{cy}}, \quad \log\Psi_\theta(y,x) = -\frac{\gamma}{cy}.$$

It is also easy to see A6.

Example 3 *For SDE (8), we assume*

$$c(x,\theta) = \frac{1}{c}, \quad f_\theta(z) = \frac{\alpha^\beta}{\Gamma(\beta)} z^{\beta-1} e^{-\alpha z},$$

and $\theta = (c, \alpha, \beta)$*, where* $c < c_0,\ \alpha > 0,\ \beta > 4$*. Then*

$$\Psi_\theta(y,x) = \frac{\alpha^\beta}{\Gamma(\beta)} (cy)^{\beta-1} e^{-\alpha c y},$$

$$\log \Psi_\theta(y,x) = \beta \log \alpha - \log \Gamma(\beta) + (\beta - 1) \log cy - \alpha c y.$$

If β *is known, then*

$$|\partial_\theta^{k+l} \Psi_\theta(y,x)|, \quad |\partial_x \partial_\theta^k \Psi_\theta(y,x)|, \quad |\partial_y \partial_\theta^k \Psi_\theta(y,x)|,$$

are all dominated by $C\,(1 + |\log y|)\, y^{\beta'} e^{-\alpha' y}$ *for some* $C > 0,\ \beta' > 3,\ \alpha' > 0$*, so from (2) to (3) are satisfied. Moreover we obtain*

$$\sup_\theta \left| \partial_\theta^{k+l} \log \Psi_\theta(y,x) \right| \leq C\,(1 + |y| + |\log y|)\,.$$

2 The construction of the contrast function

For $\dfrac{2}{\gamma+1} \leq \rho < \dfrac{1}{2}$, we define the contrast function $l_n(\alpha)$ as follows **.**

$$l_n(\alpha) = \bar{l_n}(\theta,\sigma) + \tilde{l_n}(\theta),$$

where

$$\begin{aligned}
\bar{l_n}(\alpha) &= -\frac{1}{2h_n}\sum_{i=1}^{n}(\bar{X}_{i,n})^*(\theta)\beta_{i-1}^{-1}(\sigma)\bar{X}_{i,n}(\theta)\mathbf{1}_{\{|\Delta X_i^n|\leq h_n^\rho\}} \\
&\quad -\sum_{i=1}^{n}\frac{1}{2}\log\det\beta_{i-1}(\sigma)\mathbf{1}_{\{|\Delta X_i^n|\leq h_n^\rho\}}, \\
\tilde{l_n}(\theta) &= \sum_{i=1}^{n}\Big\{\log\Phi_n(\theta, X_{t_{i-1}^n}, \Delta X_i^n)\Big\}\varphi_n(X_{t_{i-1}^n}, \Delta X_i^n)\mathbf{1}_{\{|\Delta X_i^n|>h_n^\rho\}} \\
&\quad - h_n\sum_{i=1}^{n}\int\Phi_n(\theta, X_{t_{i-1}^n}, y)\,dy,
\end{aligned}$$

$$\bar{X}_{i,n}(\theta) = X_{t_i^n} - X_{t_{i-1}^n} - h_n a_{i-1}(\theta), \quad \Phi_n(\theta,x,y) = \Psi_\theta(y,x)\varphi_n(x,y),$$

3 Asymptotic properties

Theorem 1 (Yasutaka Shimizu and Yoshida 2002) *Under the assumption from A1 to A10 and if* $nh_n \to \infty$ *as* $n \to \infty$, *the statistic* $\hat{\alpha}_n$ *which satisfies*

$$l_n(\hat{\alpha}_n) = \sup_{\alpha \in \Xi} l_n(\alpha)$$

has the consistency for the true value:

$$\hat{\alpha}_n \xrightarrow{P} \alpha_0 \quad (n \to \infty).$$

If, in addition, $nh_n^2 \to 0$ *and the true value* α_0 *is in the interior of* Ξ ,

$$\left(\sqrt{nh_n}(\hat{\theta}_n - \theta_0), \sqrt{n}(\hat{\sigma}_n - \sigma_0)\right) \xrightarrow{d} \mathcal{N}(0, K^{-1}),$$

where

$$K := \begin{pmatrix} K_1 & \mathbf{0} \\ \mathbf{0} & K_2 \end{pmatrix},$$

$$\begin{aligned} K_1^{(p,q)} &= \int (\partial_{\theta_p} a)^* \beta^{-1} (\partial_{\theta_{p'}} a)(x, \alpha_0)\, d\pi + \iint \frac{\partial_{\theta_p} \Psi_{\theta_0} \partial_{\theta_{p'}} \Psi_{\theta_0}}{\Psi_{\theta_0}}(y, x)\, dy\, d\pi, \\ K_2^{(p,q)} &= \frac{1}{2} \int \mathrm{tr}[(\partial_{\sigma_q} \beta)\beta^{-1}] \mathrm{tr}[(\partial_{\sigma_{q'}} \beta)\beta^{-1}](x, \sigma_0)\, d\pi. \end{aligned}$$

4 Comments

1. Here we discussed a "maximum likelihood" type estimator. To attain the natural asymptotic variance, we put a condition to the Lévy measure near zero. However, if we consider a particular M-estimator, it is possible to treat explosive Lévy measures. A similar methodology works.

2. Here the singular point of f_θ was only the origin. Generalization to the case where it has a singular set is possible.

3. Asymptotic expansion will be presented elsewhere. The general theory of asymptotic expasion for ϵ-Markov processes [Kusuoka and Yoshida (2000), Yoshida (2001), Sakamoto and Yoshida (1997$\sim$)] or its version can apply to discrete sampling schemes.

5 Expansion for mixing processes: local approach

There exists a general expansion theory for ϵ-Markov processes with mixing property. Indeed, expansions appearing in the following sections were obtained by applying this theory. However, in this talk, we will devote our attention to stochastic differential equations than such a general setting.

5.1 Expansion for moment estimator by Kutoyants and Y [13]

Consider a stationary diffusion process Y_t satisfying the stochastic differential equation

$$dY_t = S(Y_t)\,dt + \sigma(Y_t)\,dW_t, \quad 0 \le t \le T \tag{9}$$

and an nonparametric estimator for the expectation $\theta = E[F(Y_0)]$. Then

$$\hat{\theta}_T = \frac{1}{T}\int_0^T F(Y_t)dt$$

is asymptotically normal and asymptotically efficient in the nonparametric situation.

Theorem 2 *Let $k \in \mathbf{N}$, and let $M, \gamma, K > 0$. Suppose that $F : \mathbf{R} \to \mathbf{R}$ is not a constant. Then*

(1) there exist constants $\delta > 0$ and $c > 0$ such that for $h \in \mathcal{E}(M, \gamma)$,

$$\left|\mathbf{E}\left[h\left(\sqrt{T}\left(\hat{\theta}_T - \theta\right)\right)\right] - \Psi_{T,k}[h]\right| \;\le\; c\omega(h, T^{-K}) + \epsilon_T^{(k)},$$

where $\epsilon_T^{(k)} = o(T^{-(k+\delta)/2})$. Here $\mathcal{E}(M, \gamma) = \{h : \mathbf{R} \to \mathbf{R}$, measurable, $|h(x)| \le M(1 + |x|)^\gamma$ $(x \in \mathbf{R})$.

(2) The signed-measure $d\Psi_{T,1}$ has a density $d\Psi_{T,1}(z)/dz = p_{T,1}(z)$ with

$$p_{T,1}(z) \;=\; \phi(z; 0, \kappa_T^{(2)})\left(1 + \frac{1}{6}\kappa_T^{(3)} h_3\left(z; \kappa_T^{(2)}\right)\right),$$

where $\kappa_T^{(r)}$ is the r-th cumulant of $\sqrt{T}\left(\hat{\theta}_T - \theta\right)$. Moreover, if $q, [q]^2 - < [q]^2, f >$ belongs to $\mathcal{C}$, then

$$p_{T,1}(z) \;=\; p^*_{T,1}(z) + R_T(z),$$

where

$$p^*_{T,1}(z) \;=\; \phi(z; 0, I_*^{-1})(1 + \frac{1}{2\sqrt{T}}\mathbf{E}\left[\left[[q]^2\right][q](\xi)\right] h_3(z; I_*^{-1}))$$

and

$$\lim_{T\to\infty} \sqrt{T} \sup_{z \in \mathbf{R}} \left\{|R_T(z)| \exp(bz^2)\right\} \;=\; 0$$

for some positive constant b. In particular,

$$|\mathbf{E}\left[h\left(\sqrt{T}\left(\hat{\theta}_T - \hat{\theta}\right)\right)\right] - \int_{\mathbf{R}} h(z) p^*_{T,1}(z) dz| \;\le\; c\omega(h, T^{-K}) + \tilde{\epsilon}_T$$

for any $h \in \mathcal{E}(M, \gamma)$, with $\tilde{\epsilon}_T = o(1/\sqrt{T})$.

5.2 Asymptotic expansion for stochastic differential equations

SDE with jumps:

$$
\begin{aligned}
Y_t &= Y_0 + \int_0^t A(Y_{s-})ds + \sum_{\beta=1}^{r} \int_0^t B_\beta(Y_{s-})dw_s^\beta \\
&\quad + \sum_{\gamma=0}^{m} \int_0^t \int_{E_\gamma} C_\gamma(Y_{s-}, v^\gamma)\tilde{\mu}(ds, dv^\gamma) \\
Z_t &= Z_0 + \int_0^t A'(Y_{s-})ds + \sum_{\beta=1}^{r} \int_0^t B'_\beta(Y_{s-})dw_s^\beta \\
&\quad + \sum_{\gamma=0}^{m} \int_0^t \int_{E_\gamma} C'_\gamma(Y_{s-}, v^\gamma)\tilde{\mu}(ds, dv^\gamma).
\end{aligned}
$$

Coefficients may degenerate.

Note: The nonparametric estimation problem in 5.1 is an example.

Assumptions.
[$A1$] Mixing condition for Y.
[$A2$] Moment condition.
[$A3$] Malliavin calculus: local nondegeneracy of $(Y_v, Z_v - Z_u)$ given Y_u.

Theorem. (Kusuoka and Y., Y.)
$[A1]$ + $[A2]$ + $[A3]$ $\Rightarrow$ asymptotic expansion of $E[f(Z_T/\sqrt{T})]$ for measurable f.

Asymptotic expansion of $E[f(Z_T/\sqrt{T})]$

Let $\Delta_T(f) := \left|P\left[f\left(\frac{1}{\sqrt{T}}Z_T\right)\right] - \Psi_{T,p}\left[f\right]\right|$. We want to obtain the following inequality:

$$\Delta_T(f) \;\leq\; M^*[\omega_2\left(f; T^{-K}, \phi(x;0,u_T I_d)dx\right) + \bar{o}(T^{-(p-2+\delta^*)/2})].$$

5.3 How to verify Condition $[A3]$ corresponding to the "conditional type Cramér condition"?

1. Hörmander condition
2. Other nondegeneracy condition for densities
3. Local nondegeneracy with support theorems

An application to a financial model

(with Hiroki Masuda) Let X_t be a positive stationary Ornstein-Uhlenbeck process driven by a Lévy process Z_t, that is, the following stochastic differential equation is satisfied:

$$dX_t \;=\; -\lambda X_t dt + dZ_t.$$

Furthermore, we assume the strong mixing property with exponential decay for X_t. Barndorff-Nielsen and Shephard (JRSS 2001) discussed a financial stochastic volatility model

$$dS_t \;=\; (\mu + \beta X_t)dt + \sqrt{X_t} dw_t.$$

Here S_t denotes the logarithm of the stock price at time t. The log-return of the stock price in the n-th period of length Δ is given by

$$\tilde{S}_n(\Delta) \;=\; S_{n\Delta} - S_{(n-1)\Delta}.$$

Let $H_\Delta = \tilde{S}_1(\Delta) - E[\tilde{S}_1(\Delta)]$. Obviously, for each n, $H_\Delta/\sqrt{\Delta}$ asymptotically has a normal distribution when $\Delta \to \infty$. Applying the previous result, we obtain an expansion of $\mathcal{L}\{H_\Delta/\sqrt{\Delta}\}$. The first-order asymptotic expansion is given by the density

$$p_3(z) \;=\; \phi(z;\Sigma_\Delta)\left[1 + 6^{-1}\chi_{3,\Delta}(z^3/\Sigma_\Delta^3 - 3z/\Sigma_\Delta^2)\right]$$

and $\chi_{3,\Delta} = O(\Delta^{-1/2})$ has an explicit expression.

6 Sampling Problems: Small History and Prescriptions for Expansions

Diffusion model.

$$
\begin{aligned}
dX_t &= V_0(X_t, \theta_1)dt + \epsilon V(X_t, \theta_2)dw_t, \quad t \in [0, T] \\
X_0 &= x_0,
\end{aligned}
$$

$\epsilon = 1$ or $\epsilon \downarrow 0$.

Data: $\{X_{ih}\}_{i=0,1,\ldots,n}$.

6.1 $(h \to 0)$+$(T = hn \to \infty)$ case

- The first-order asymptotic properties were discussed in Prakasa-Rao [24, 25], Yoshida [47], Kessler [8].
- Shimizu and Yoshida [32] presented an estimating functional for a stochastic differential equation with jumps. The first-order properties have been proved in it.
- The asymptotic expansions will follow from a version of the local approach (Yoshida [54] and Sakamoto and Yoshida [30]).

6.2 (fixed h)+($T = hn \to \infty$) case

- This case was treated in Bibby and Sørensen [1].
- Masuda [14, 15] proved the asymptotic normality of the moment estimator for a state space model involving jump noise terms.
- In each case, the asymptotic expansion will follow as in Section 6.1.

6.3 ($h \to 0$)+(fixed $T = hn$) case

- The first-order asymptotics for the diffusion coefficient was obtained by Genon-Catalot and Jacod [5].
- For the linearly parameterized case, the expansion of the estimator was derived in Yoshida [51].
- General case is open.

6.4 (fixed h)+(fixed T)+(small diffusion) case

- The first-order inference was discussed in Sørensen [33].
- Asymptotic expansion can be deduced from the global approach (Sakamoto and Yoshida [29]) or from a version of Yoshida [50].

6.5 $(h \to 0)$+(fixed T)+(small diffusion) case

- The first-order asymptotic results were obtained by Sørensen and Uchida [34], Uchida [39, 40].
- Local asymptotic normality was proved by Uchida [41].
- The martingale expansion method ([51, 53]) can apply to derive asymptotic expansions.
- Small dispersion parameter (Condition $\epsilon \downarrow 0$) can be removed to keep identifiability in the limit by making a Berry-Esseen type bound which one obtains by a similar technique to the martingale method.

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