ADAPTIVE PROJECTION ESTIMATORS FOR A CLASS OF FUNCTIONAL PARAMETER

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PLAN

I - A CLASS OF FUNCTIONAL PARAMETERS

- $\mathcal{P}$ family of stochastic processes with values in $(E, \mathcal{B})$.

  $X = (X_t, t \in \mathbb{Z}) \in \mathcal{P}$

PARAMETER $\boxed{\varphi_X = g(P_X)}$

$\varphi = \varphi_X \in \Phi \subset H$, HILBERT SPACE
SEPARABLE, $\|\cdot\|$, $\langle \cdot , \cdot \rangle$

- φ e-adapted

  $e = (e_j, j \in J)$ ORTHONORMAL SYSTEM OF H (for example $J = \mathbb{N}$)

$$\boxed{\varphi = \sum_{j=0}^{\infty} \varphi_j e_j}$$

$$\boxed{\varphi_j = E\left[h_j(X_0, \ldots, X_{\nu(j)})\right]}$$

$$\nu(j) \leq j$$

EXAMPLES

1. SPECTRAL DENSITY

$$\varphi(\lambda) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} E(X_0 X_j) \cos \lambda j$$

$, \lambda \in [-\pi, +\pi]$

2. DISTRIBUTION FUNCTION (in some situations)
3. DENSITY
4. DERIVATIVES OF DENSITY
5. REGRESSION (Density known)
6. DISCRETE DISTRIBUTION
7. COVARIANCE OPERATOR

X STATIONARY, VALUES IN G HILBERT

$E X_0 = 0$, $E \|X_0\|^4 < \infty$

$$C(x) = E\left(\langle X_0, x \rangle_G X_0 \right), \; x \in G$$

$C \in H = \mathcal{S}_G$ SPACE OF HILBERT-SCHMIDT OPERATORS ON G

$$C = \sum_{j=0}^{\infty} E(\langle X_0, v_j \rangle^2)\, v_j \otimes v_j$$

8. CROSS COVARIANCE OPERATORS

$$C_{X_0 X_1} = \sum E(\langle X_0, f_j \rangle \langle X_1, g_j \rangle)\, f_j \otimes g_j$$

II - ORACLE

PROBLEM : SPECIFY ESTIMATION RATES FOR SOME SUBCLASSES OF Φ.

- SUBCLASSES OF Φ

$$\boxed{\Phi_b = \{\varphi : \varphi \in \Phi,\ |\varphi_j| \leq b_j\ ,\ j \geq 0\}}$$

$$\sum_{j=0}^{\infty} b(j) < \infty\ , \quad j\, b(j) \to 0,\ j \uparrow \infty$$

- ASSUMPTIONS

A_0 - $\exists\, \boxed{\varphi^*} \in \Phi_b,\ \varphi_j^* \neq 0\ ,\ j \geq 0$ AND

$$|\varphi_j^*| = \delta\, b_j\ ,\ j \geq j_0\ (0 < \delta \leq 1)$$

(NATURAL !)

$\longrightarrow$

$\boxed{A_1}$ - SET, FOR $n > \tau(j)$,

$$\boxed{\overline{\varphi}_{jn} = \frac{1}{n-\tau(j)} \sum_{i=1}^{n-\tau(j)} h_j(X_i, \dots, X_{i+\tau(j)})}$$

THEN $E_\varphi \overline{\varphi}_{jn} = \varphi_j$ AND

$$\boxed{0 \leq \frac{\alpha_j}{u_n} \leq v_{jn}^{*} \leq \frac{\beta_j}{u_n}}$$

$(u_n) \nearrow \infty$, $u_n \leq n$

WHERE $v_{jn}^{*} = \mathrm{Var}_{X^*} \overline{\varphi}_{jn}$

$\varphi^* = g(P_{X^*})$

- MOREOVER

$$\boxed{\sup_{X \in \mathcal{P}_b} \mathrm{Var}_X \overline{\varphi}_{jn} \leq c \frac{\beta_j}{u_n}}$$

—

- FAMILY OF LINEAR ESTIMATORS

$$\boxed{\bar{\varphi}_n = \sum_{j=0}^{\infty} \lambda_{jn} \bar{\varphi}_{jn} \mathbb{1}_{\{n > J(j)\}} e_j}$$

$\bar{\varphi}_n \in \mathcal{F}$

ORACLE = THE BEST ESTIMATOR AT φ^* BELONGING TO $\mathcal{F}$

$$\boxed{\varphi_n^{o} = \sum_{j=0}^{\infty} \frac{\varphi_j^{*2}}{\varphi_j^{*2} + V_{jn}^{*}} \bar{\varphi}_{jn} \mathbb{1}_{\{n > J(j)\}} e_j}$$

THE ORACLE FURNISHES THE MINIMAX RATE ($/\mathcal{F}$)

III. RATES

THEOREM 1 (PARAMETRIC RATE)

IF A_0, A_1 HOLD WITH

$$\boxed{0 < \sum_j \alpha_j \leq \sum_j \beta_j < \infty}$$

THEN, FOR $\lambda(j) = 0,\ j \geq 0,$

$$\boxed{\underline{\lim}_{n\to\infty} \inf_{\bar{\varphi}_n \in \mathcal{F}} \sup_{X \in \mathcal{P}_b} u_n . E_X \|\bar{\varphi}_n - \varphi_X\|^2 \geq \sum_j \alpha_j}$$

SET

$$\boxed{\tilde{\varphi}_n = \sum_{j=0}^{u_n} \bar{\varphi}_{jn} \, e_j}$$

THEN

$$\boxed{\sup_{X \in \mathcal{P}_b} u_n . E_X \|\tilde{\varphi}_n - \varphi_X\|^2 \leq c \sum_j \beta_j}$$

$\longrightarrow$ RATE $\boxed{\dfrac{1}{u_n}}$

NONPARAMETRIC RATES

Definition

A STRICTLY POSITIVE REAL SEQUENCE WILL BE SAID TO BE OF MODERATE VARIATION (MV)

IF $$\boxed{\bar{\gamma}_k \simeq \gamma_k}$$

WHERE $$\bar{\gamma}_k = \frac{1}{k} \sum_0^k \gamma_j$$

Properties

$$\sum_0^\infty \gamma_j = \infty \quad , \quad \gamma_k \simeq \gamma_{k+1}$$

example

$$\gamma_j = c\, j^a \left[\log(j+1)\right]^b$$

$$c > 0,\ a > -1,\ b \in \mathbb{N}$$

ASSUMPTION

$$\boxed{A_2 - (\beta_j) \text{ IS MV AND } \sum_0^k \alpha_j \simeq \sum_0^k \beta_j}$$

NOW ONE DEFINES k_m^* BY

$$\boxed{\varphi_{k_m^*}^{*2} \geq d^2 \frac{\beta_{k_m^*}}{u_m}}$$

"OPTIMAL TRUNCATION INDEX"

AND

$$\boxed{\varphi_j^{*2} < d^2 \frac{\beta_j}{u_m}, \; j > k_m^*}$$

$$\boxed{A_3 - \beta_j \varphi_j^{*-2} \nearrow; \; \beta_{k_m^*} \geq \gamma \beta_{k_m^*+1} \; (\gamma > 0)}$$

technical but mild conditions!

LEMMA (ORACLE)

[redacted] $$\sum_{j>k_m^*} \varphi_j^{*2} = O\left(\sum_0^{k_m^*} \beta_j \cdot u_m^{-1}\right) = O(v_m$$

$$\boxed{E_{X^*} \|\varphi_m^0 - \varphi^*\|^2 \simeq v_m}$$

THEOREM 2 (NONPARAMETRIC RATE)

IF A_0, A_1, A_2, A_3 HOLD

$$\boxed{\underline{\lim}_{m \to \infty} \inf_{\bar{\varphi}_m \in \mathcal{F}} \sup_{X \in \mathcal{G}_b} v_m^{-1} \cdot E_X \|\bar{\varphi}_m - \varphi_X\|^2 > 0}$$

SET

$$\boxed{\hat{\varphi}_m = \sum_{j=0}^{k_m^*} \overline{\varphi_{jn}}\, e_j}$$

THEN

$$\boxed{\sup_{X \in \mathcal{G}_b} E_X \|\hat{\varphi}_m - \varphi\|^2 = O(v_m)}$$

IV - APPLICATIONS

PARAMETRIC RATE

Distribution function
Discrete distribution
Covariance operator
Cross Covariance operator

NONPARAMETRIC RATE

Density
Derivatives of density
Regression
Spectral density

EXAMPLE : SPECTRAL DENSITY

$$\varphi(\lambda) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} E(X_0 X_j) \cos \lambda j, \quad |\lambda| \leq \pi$$

IF $\underline{b_j = a r^j}$ $(a > 0,\ 0 < r < 1)$

$\mathcal{P}_b$ CONTAINS LINEAR PROCESSES OF THE FORM

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \quad t \in \mathbb{Z}$$

WITH

$$|a_j| \leq c \rho^j \qquad (c > 0,\ 0 < \rho \leq r)$$

HERE φ^* IS ASSOCIATED WITH A AR(1)-PROCESS

$$V_{jn}^* \simeq \frac{1+r^2}{1-r^2} \cdot \frac{1}{m} \simeq \frac{\beta_j}{m}$$

$$k_m^* \simeq \log m$$

AND THE OPTIMAL RATE IS

$$\boxed{\frac{\log m}{m}}$$

NOT A SURPRISE !

V - ADAPTIVE ESTIMATORS

IN PRACTICE k_n^* IS UNKNOWN

$\Rightarrow$ TRUNCATION INDEX BASED ON THE DATA : $\hat{k}_n$

$$\boxed{\hat{\varphi}_n = \sum_{j=0}^{\hat{k}_n} \overline{\varphi}_{jn} \, e_j}$$

$$\boxed{\hat{k}_n = \max \{ j : 0 \le j \le k_n , |\overline{\varphi}_{jn}| \ge \gamma_n \}}$$

k_n, γ_n GIVEN

($\hat{k}_n = k_n$ IF $\{ \dots \} = \emptyset$)

$$k_n \to \infty, \quad \frac{k_n}{n} \to 0, \quad \gamma_n \to 0$$

EXAMPLE

THE CASE OF SPECTRAL DENSITY

- IF (X_t) IS A <u>MOVING AVERAGE OF ORDER K</u> UNKNOWN, AND φ ITS SPECTRAL DENSITY, THEN THE CHOICE

$$\boxed{\gamma_m = \frac{(\log m . \log \log m)^{1/2}}{m^{1/2}}}$$

LEADS TO

$$\boxed{\hat{k}_m = K}$$ A.S. m LARGE ENOUGH

AND THE RATE IS $\boxed{\boxed{1/m}}$

→ DOUBLE ESTIMATION

- IN THE <u>GENERAL CASE</u>, UNDER SOME MIXING CONDITIONS THE RATE IS <u>OPTIMAL UP TO A LOGARITHM.</u>

VI - ADAPTIVE ESTIMATORS IN CONTINUOUS TIME

(THE CASE OF DENSITY)

$$(\Omega, \mathcal{A}, P) \xrightarrow[X_t]{} (E, \mathcal{B}, \lambda)$$

σ-FINITE

$(X_t, t \in \mathbb{R})$ MEASURABLE

$X_t \sim f \cdot \lambda \quad , \quad t \in \mathbb{R}$

$$\Phi = \left\{ f : f = \sum_{j=0}^{\infty} \varphi_j e_j \ , \ \sum \varphi_j^2 < \infty \right\}$$

(e_j) ORTHONORMAL SYSTEM IN $L^2(\lambda)$

DATA : $\boxed{X_t, \ 0 \leq t \leq T}$

$$\boxed{\hat{f}_T = \sum_{j=0}^{\hat{k}_T} \underbrace{\left[\frac{1}{T} \int_0^T e_j(X_t)\, dt \right]}_{\text{" } \hat{a}_{jT}} e_j}$$

$$\hat{k}_T = \max \{ j : 0 \leq j \leq k_T, \ |\hat{a}_{jT}| \geq \gamma_T \}$$

WHERE

$$\boxed{\gamma_T = \frac{\log T \, (\log \log T)^{1/2}}{T^{1/2}}}$$

UNDER REGULARITY ASSUMPTIONS
(EX: STATIONARY <u>DIFFUSION</u> PROCESS)

- IF $\varphi_j = 0 \ , \ j > K$

THEN $\hat{k}_T = K$ (A.S. T large)

AND

$$\boxed{T \cdot E\|\hat{f}_T - f\|^2 \longrightarrow 2\sum_{j=0}^{K}\int_0^\infty \mathrm{Cov}\left(e_j(X_0), e_j(X_u)\right) du}$$

- IF $|\varphi_j| = O(\rho^{j})$ $0<\rho<1$
(FOR EXAMPLE)

THEN

$$\boxed{E\|\hat{f}_T - f\|^2 = O\left(\frac{(\log T)^2 \log\log T}{T}\right)}$$

- IN SOME SITUATIONS $\hat{f}_T$ IS BETTER THAN THE <u>LOCAL TIME</u> ESTIMATOR

CF $f_{0,T} = \sum_0^\infty \hat{a}_{jT}\, e_j$

REFERENCES

IGNACCOLO (2002), AUBIN (2004) PREPRINT
(TEST) (MIXING CASE) (REGRESSION)

CARBON (1982) PHD THESIS
(POINTWISE CONVERGENCE)

BLANKE, BOSQ (2004)
LOCAL SUPEREFFICIENCY OF DATA-DRIVEN PROJECTION DENSITY ESTIMATORS IN CONTINUOUS TIME (SORT)

BOSQ (2005)
INFÉRENCE ET PRÉVISION EN GRANDES DIMENSIONS (ECONOMICA)
(DENSITY ONLY)

BOSQ (2005)
ESTIMATION SUROPTIMALE DE LA DENSITÉ PAR PROJECTION (CANAD. J)

BLANKE, BOSQ (2007)
INFERENCE AND PREDICTION IN LARGE DIMENSIONS (WILEY-DUNOD)
300 [redacted] P.
(GENERAL CASE)