

ADAPTIVE PROJECTION ESTIMATORS FOR A CLASS OF FUNCTIONAL PARAMETER

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PLAN

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TIME

I. A CLASS OF FUNCTIONAL PARAMETERS

- \mathcal{P} family of stochastic processes with values in (E, \mathcal{B}) .
 $X = (X_t, t \in \mathbb{Z}) \in \mathcal{P}$

PARAMETER $\varphi_X = g(P_X)$

$\mathcal{P} = \varphi_X \in \Phi \subset H$, HILBERT SPACE
SEPARABLE, $\|\cdot\|$, $\langle \cdot, \cdot \rangle$

- φ e-adapted
 $e = (e_j, j \in J)$ ORTHONORMAL SYSTEM OF H (for example $J = \mathbb{N}$)

$$\varphi = \sum_{j=0}^{\infty} \varphi_j e_j$$

$$\varphi_j = E [h_j(X_0, \dots, X_{2(j)})]$$

$$2(j) \leq j$$

EXAMPLES

1. SPECTRAL DENSITY

$$\varphi(\lambda) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} E(X_0 X_j) \cos \lambda j$$

$$\lambda \in [-\pi, +\pi]$$

2. DISTRIBUTION FUNCTION (in some situations)

3. DENSITY

4. DERIVATIVES OF DENSITY

5. REGRESSION (Density known)

6. DISCRETE DISTRIBUTION

7. COVARIANCE OPERATOR

X STATIONARY, VALUES IN G HILBERT

$$E X_0 = 0, \quad E \|X_0\|^4 < \infty$$

$$C(\alpha) = E(\langle X_0, \alpha \rangle_G X_0), \quad \alpha \in G$$

$C \in H = \int_G$ SPACE OF HILBERT-SCHMIDT OPERATORS ON G

$$C = \sum_{j=0}^{\infty} E(\langle X_0, \nu_j \rangle^2) \nu_j \otimes \nu_j$$

8. CROSS COVARIANCE OPERATORS

$$C_{X_0, X_1} = \sum_j E(\langle X_0, f_j \rangle \langle X_1, g_j \rangle) f_j \otimes g_j$$

II. ORACLE

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PROBLEM : SPECIFY ESTIMATION RATES
FOR SOME SUBCLASSES OF Φ .

• SUBCLASSES OF Φ

$$\Phi_b = \{ \varphi : \varphi \in \Phi, |\varphi_j| \leq b_j, j \geq 0 \}$$

$$\sum_{j=0}^{\infty} b(j) < \infty, \quad j b(j) \rightarrow 0, j \uparrow \infty$$

• ASSUMPTIONS

A₀ $\exists \varphi^* \in \Phi_b, \varphi_j^* \neq 0, j \geq 0$ AND

$$|\varphi_j^*| = \delta b_j, j \geq j_0 \quad (0 < \delta \leq 1)$$

(NATURAL !)



A_1 - SET, FOR $m > 2(j)$,

$$\bar{\varphi}_{ijn} = \frac{1}{m-2(j)} \sum_{i=1}^{m-2(j)} h_j(x_i, \dots, x_{i+2(j)})$$

THEN $E \bar{\varphi}_{ijn} = \varphi_j$ AND

$$0 \leq \frac{\alpha_j}{\mu_m} \leq V_{ijn}^* \leq \frac{\beta_j}{\mu_m}$$

$(\mu_m) \rightarrow \infty$, $\mu_m \leq m$

WHERE

$$V_{ijn}^* = \text{Var}_{X^*} \bar{\varphi}_{ijn}$$

$$\varphi^* = g(P_{X^*})$$

• MOREOVER

$$\sup_{X \in \mathcal{D}_b} \text{Var}_X \bar{\varphi}_{ijn} \leq c \frac{\beta_j}{\mu_m}$$

• FAMILY OF LINEAR ESTIMATORS

$$\bar{\varphi}_m = \sum_{j=0}^{\infty} \lambda_{jm} \bar{\varphi}_{jm} \mathbb{1}_{\{m > \omega(j)\}} e_j$$

$$\bar{\varphi}_m \in \mathcal{F}$$

ORACLE = THE BEST ESTIMATOR
AT φ^* BELONGING TO \mathcal{F}

$$\varphi_m^0 = \sum_{j=0}^{\infty} \frac{\varphi_j^{*2}}{\varphi_j^{*2} + V_{jm}^*} \bar{\varphi}_{jm} \mathbb{1}_{\{m > \omega(j)\}} e_j$$

THE ORACLE FURNISHES

THE MINIMAX RATE $(/\mathcal{F})$

III. RATES

THEOREM 1 (PARAMETRIC RATE) 6

IF A_0, A_1 HOLD WITH

$$0 < \sum_j \alpha_j \leq \sum_j \beta_j < \infty$$

THEN, FOR $\alpha(j) = 0, j \geq 0,$

$$\lim_{m \rightarrow \infty} \inf_{\bar{\varphi}_m \in \mathcal{F}} \sup_{X \in \mathcal{P}_b} u_m \cdot E_X \|\bar{\varphi}_m - \varphi_X\|^2 \geq \sum_j \alpha_j$$

SET

$$\tilde{\varphi}_m = \sum_{j=0}^{u_m} \bar{\varphi}_{jm} e_j$$

THEN

$$\sup_{X \in \mathcal{P}_b} u_m \cdot E_X \|\tilde{\varphi}_m - \varphi_X\|^2 \leq C \sum_j \beta_j$$

→ RATE

$$\boxed{\frac{1}{u_m}}$$

NONPARAMETRIC RATES

Definition

A STRICTLY POSITIVE REAL SEQUENCE WILL BE SAID TO BE OF MODERATE VARIATION (MV)

IF

$$\bar{\gamma}_k \approx \gamma_k$$

WHERE

$$\bar{\gamma}_k = \frac{1}{k} \sum_0^k \gamma_i$$

Properties

$$\sum_0^{\infty} \gamma_i = \infty, \quad \gamma_k \approx \gamma_{k+1}$$

example

$$\gamma_j = c j^a [\log(j+1)]^b$$

$$c > 0, a > -1, b \in \mathbb{N}$$

ASSUMPTION

$$A_2 - (\beta_j) \text{ is MV AND } \sum_0^k \alpha_j \approx \sum_0^k \beta_j$$

NOW ONE DEFINES k_m^* BY

$$\varphi_{k_m^*}^{*2} \geq d^2 \frac{\beta_{k_m^*}}{u_m}$$

"OPTIMAL TRUNCATION INDEX"

AND

$$\varphi_j^{*2} < d^2 \frac{\beta_j}{u_m}, \quad j > k_m^*$$

$$A_3 - \beta_j \varphi_j^{*-2} \uparrow; \quad \beta_{k_m^*} \geq \gamma \beta_{k_m^*+1} \quad (\gamma > 0)$$



Technical but mild conditions!

LEMMA (ORACLE)

$$\sum_{j > k_m^*} \varphi_j^{*2} = O\left(\frac{k_m^*}{0} \beta_j \cdot u_m^{-1}\right) = O(u_m)$$

$$E_{x^*} \|\varphi_m^0 - \varphi^*\|^2 \approx u_m$$

THEOREM 2 (NONPARAMETRIC RATE)

IF A_0, A_1, A_2, A_3 HOLD

$$\lim_{m \rightarrow \infty} \inf_{\bar{\varphi}_m \in \mathcal{F}} \sup_{x \in \mathcal{D}_x} u_m^{-1} E_x \|\bar{\varphi}_m - \varphi_x\|^2 > 0$$

SET

$$\hat{\varphi}_m = \sum_{j=0}^{k_m^*} \varphi_{jn} e_j$$

THEN

$$\sup_{x \in \mathcal{D}_x} E_x \|\hat{\varphi}_m - \varphi\|^2 = O(u_m)$$

IV. APPLICATIONS

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PARAMETRIC RATE

Distribution function
Discrete distribution
Covariance operator
Cross Covariance operator

NONPARAMETRIC RATE

Density
Derivatives of density
Regression
Spectral density

EXAMPLE: SPECTRAL DENSITY 11

$$\varphi(\lambda) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} E(X_0 X_j) \cos \lambda j, |\lambda| \leq \pi$$

IF $b_j = a r^j$ ($a > 0, 0 < r < 1$)

\mathcal{S}_b CONTAINS LINEAR PROCESSES
OF THE FORM

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, t \in \mathbb{Z}$$

WITH

$$|a_j| \leq c r^j \quad (c > 0, 0 < r \leq 1)$$

HERE φ^* IS ASSOCIATED WITH
A AR(1)-PROCESS

$$V_{jn}^* \approx \left(\frac{1+r^2}{1-r^2} \right) \cdot \frac{1}{n} \approx \frac{\beta_j}{n}$$

$$R_m^* \approx \log_2 m$$

AND THE OPTIMAL RATE IS

$$\boxed{\frac{\log_2 m}{n}}$$

NOT A
SURPRISE!

