Estimation for the standard and geometric telegraph process

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1. Telegraph process

Consider a particle moving on the real line starting at the origin at time $t = 0$ with initial velocity $+v$ (rightward) or $-v$ (leftward) chosen at random with equal probability.

The particle continues to move in one direction for an exponential time, then the velocity is reversed.

The *telegraph process* $\{X(t), t \geq 0\}$ describes the position of the particle at time t

1. Telegraph process

More precisely, the *telegraph process* (Goldstein, 1951 and Kac, 1974) is defined as follows

$$X(t) = V(0)\int_0^t (-1)^{N(s)} \mathrm{d}s$$

where $V(0) =$ is the random initial velocity such that $P(V(0) = +v) = P(V(0) = -v) = \frac{1}{2}$ and $V(0)$ is independent of the Poisson process $\{N(t), t \geq 0\}$ with constant rate λ

1. Telegraph process

The process $\{X(t), t \geq 0\}$ has continuous paths and finite velocity. Hence it can describe motions of bodies which posses inertia.

The process $X(t)$ takes values in the bounded interval $[-vt, +vt]$. Indeed, if one or more events occur in the time $[0, t]$ then $|X(t)| < vt$.

$V(t) = V(0)(-1)^{N(t)}$ is called the velocity process

1. Telegraph equation

Denote by $p(t,x)\mathrm{d}x = P(X(t) \in [x, x + \mathrm{d}x))$ the density of the distribution of the particle at time t.

$X(t)$ is called the *telegraph* (or *telegrapher's*) process because $p(t,x)$ solves the following so-called telegraph (or wave) equation

$$\frac{\partial^2}{\partial t^2} f(t,x) + 2\lambda \frac{\partial}{\partial t} f(t,x) = v^2 \frac{\partial^2}{\partial x^2} f(t,x) \qquad (1)$$

for $p(x,0) = \delta(x)$ and $\frac{\partial}{\partial t} p(t,x)\Big|_{t=0} = 0$

Equation (1) describes the propagation of waves along strings

1. Telegraph process

The explicit expression of the density p has been found by Goldstein (1951), Orsingher (1990) and Pinsky (1991) using different techniques. It is a mixture of a discrete part:

$$P(X(t) = \pm vt) = \frac{1}{2}e^{-\lambda t}$$

and a continuous term which we present, in the next slide, in the general form for arbitrary starting point $X(t_0) = x_0$

$$p(t, x; t_0, x_0) = P(X(t) \in [x, x + \mathrm{d}x) | X(t_0) = x_0), \quad |x - x_0| < v(t - t_0)$$

1. Telegraph process

$$p(t, x; t_0, x_0) =$$

$$\frac{e^{-\lambda(t-t_0)}}{2v}\left\{\lambda I_0\left(\frac{\lambda}{v}\sqrt{u_t(x,x_0)}\right) + \frac{v\lambda(t-t_0)I_1\left(\frac{\lambda}{v}\sqrt{u_t(x,x_0)}\right)}{\sqrt{u_t(x,x_0)}}\right\}\chi_{\{u_t(x,x_0)>0\}}$$

$$+\frac{e^{-\lambda t}}{2}\delta(u_t(x,x_0))$$

where χ_A is the indicator function of set A,

$$u_t(x,x_0) = v^2(t-t_0)^2 - (x-x_0)^2$$

δ is the Dirac delta, I_k's are the modified Bessel functions of order k of first kind.

The distribution is entirely described by the parameter λ (and v)

The telegraph process $X(t)$ is not Markovian, not ergodic and not even stationary. All moments of $X(t)$ of odd order are zero, but (I. &Yoshida, 2006), for $q \geq 1$, we have

$$\mathbf{E}\{X(t)\}^{2q} = (vt)^{2q} \left(\frac{2}{\lambda t}\right)^{q-\frac{1}{2}} \Gamma\left(q + \frac{1}{2}\right) \left\{ I_{q+\frac{1}{2}}(\lambda t) + I_{q-\frac{1}{2}}(\lambda t) \right\} e^{-\lambda t}$$

In particular, (Orsingher, 1990):

$$EX^2(t) = \frac{v^2}{\lambda}\left(t - \frac{1 - e^{-2\lambda t}}{2\lambda}\right)$$

2. Why the telegraph process?

The telegraph process $X(t)$ moves at finite velocity, has finite support and finite variation and seems to be a more realistic way to describe random evolutions than usual Brownian motion.

2. Why the telegraph process?

- Di Crescenzo and Pellerey (2002) proposed the geometric telegraph process to describe the dynamics of the price of risky assets, i.e. $S(t) = s_0 \exp\{\alpha t + \sigma X(t)\}$, $t > 0$;

- Mazza and Rulliere (2004) linked the process $X(t)$ and the ruin processes in the context of risk theory;

- Di Masi *et al* (1994) proposed to model the volatility of financial markets in terms of the telegraph process;

- Ratanov (2004, 2005) proposed to model financial markets using a telegraph process with two intensities $\lambda_\pm$ and two velocities $c_\pm$;

- Holmes *et al.* (1994) and Holmes (1993) used $X(t)$ in ecology to model population dynamics and to describe the displacement of wild animals on the soil. In particular, this model is chosen because it preserves the property of animals to move at finite velocity and for a certain period along some direction.

2. Relations with Brownian motion

The telegraph equation is a partial differential equation of hyperbolic type and its relationship with stochastic processes is due to Goldstein (1951).

If $\lambda \to \infty$, $v \to \infty$ such that $v^2/\lambda \to 1$ the telegraph equation reduces to the heat equation

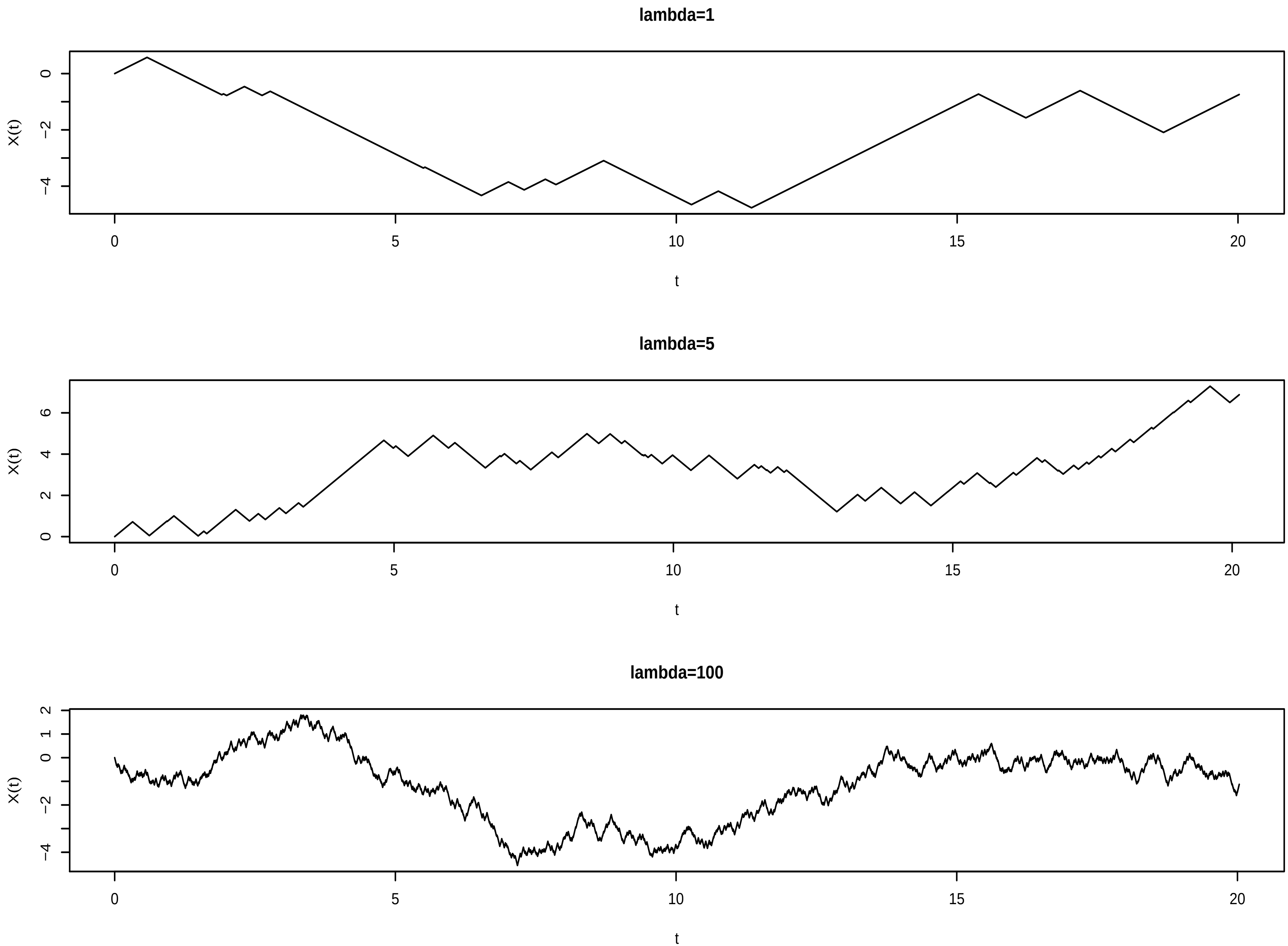
$$\frac{\partial}{\partial t} f(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t,x)$$

2. Relations with Brownian motion

The density of the distribution of the Brownian motion $B(t) \sim \mathcal{N}(0, t)$, solves the previous heat equation and, as we have just seen, in the limit as $\lambda \to \infty$, $v \to \infty$ such that $v^2/\lambda \to 1$ the telegraph equation reduces to the heat equation

This means that if we allow the telegraph process to move at infinite speed and with infinitely many changes of direction its paths converge to the paths of the Brownian motion.

Next slide is a graphical evidence of this behaviour



lambda=1
X(t)
0
−2
−4
0
5
10
15
20
t
lambda=5
X(t)
6
4
2
0
0
5
10
15
20
t
lambda=100
X(t)
2
1
0
−2
−4
0
5
10
15
20
t

3. Inference for the telegraph process

Inference for the telegraph process with continuous observation does not present difficulties because the problem is equivalent to the estimation of the rate λ of the underlying Poisson process (Kutoyants, 1998). The problem is of some interest for the non-homogeneous telegraph process.

We present estimation problems for: the non-homogeneous telegraph process with continuous observations (I. 2001); for the standard and geometric telegraph process from discrete time observations (De Gregorio & I. 2006, I. & Yoshida, 2006)

3. Non-homogeneous telegraph equation (I. 2001)

For the non-constant rate $\lambda = \lambda(t)$ the solution of the non-homogeneous telegraph equation

$$\frac{\partial^2}{\partial t^2}u(t,x) + 2\lambda(t)\frac{\partial}{\partial t}u(t,x) = v^2\frac{\partial^2}{\partial x^2}u(t,x) \qquad (2)$$

has been found, but only for the particular parametric family of λ

$$\lambda(t) = \lambda_\theta(t) = \theta \tanh(\theta t) \left(\stackrel{t\to\infty}{\longrightarrow} \theta\right), \qquad \theta \in \mathbb{R}\,.$$

Theorem: the explicit solution of (2) is

$$p_\theta(t,x) = \begin{cases} \frac{\theta\, t}{\cosh(\theta\, t)} \frac{I_1\left(\frac{\theta}{v}\sqrt{v^2\, t^2 - x^2}\right)}{2\sqrt{v^2\, t^2 - x^2}}, & |x| < vt \\ 0, & \text{otherwise} \end{cases}$$

3. Inference for the non-homogenous telegraph process

Given n copies of the telegraph process (i.e. n moving particles), consistency, asymptotic normality and asymptotic efficiency (in the minimax sense) of the estimator $\widehat{\theta}_n$ of θ in $\lambda_\theta(t) = \theta\,\mathrm{tanh}(\theta t)$ has been established, where

$$\widehat{\theta}_n = \frac{1}{T}\,\mathrm{arcosh}\left(e^{\widehat{\pi}_n}\right)$$

and $\widehat{\pi}_n$ is the average number of observed switches of direction of all particles up to time T

3. Inference for the non-homogenous telegraph process

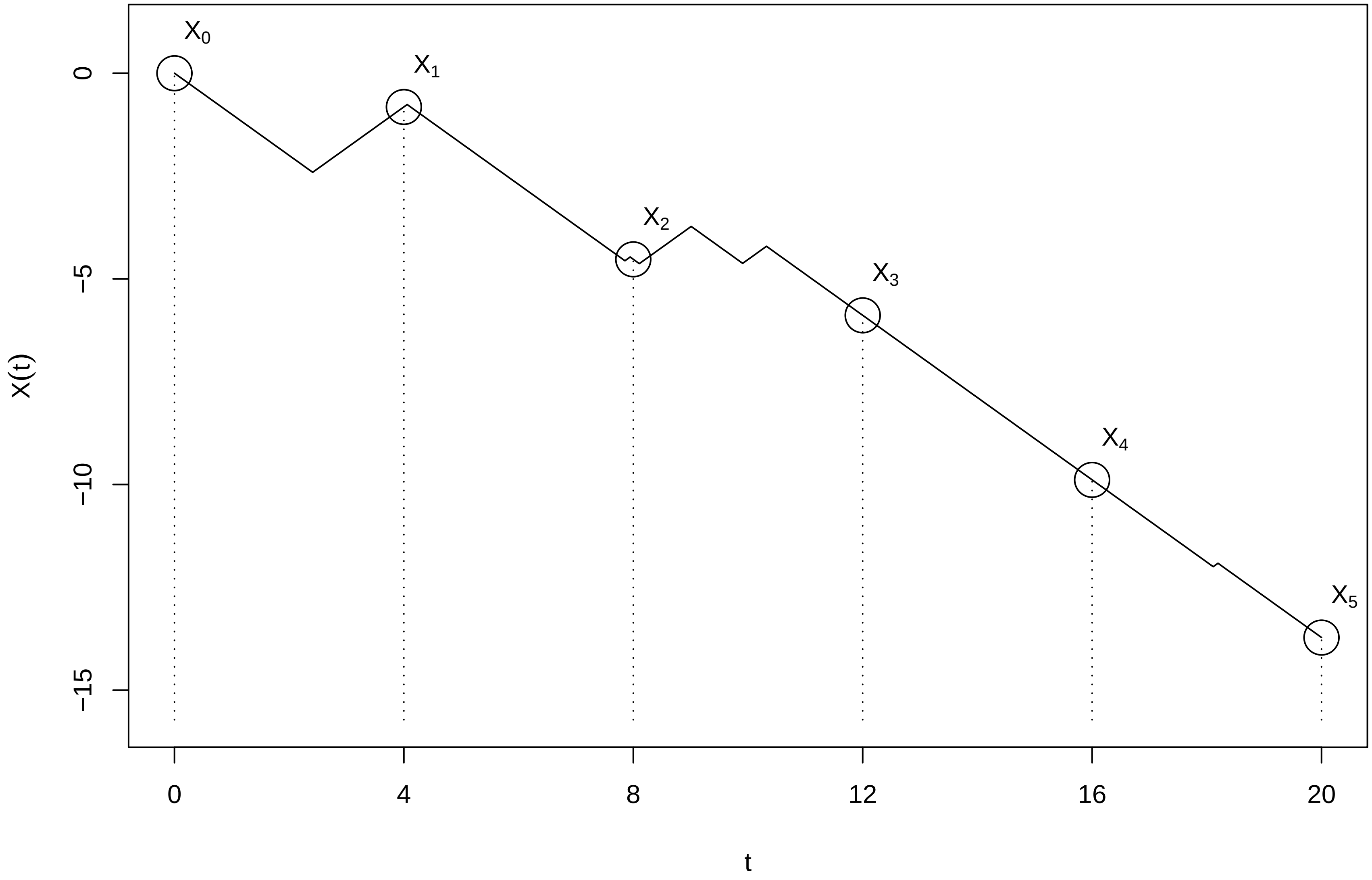
More precisely, as $n \to \infty$, i.e. as the number of observed particles increases, we obtained

$$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}\left(0, T^2 \frac{\ln(\cosh(\theta T))}{\coth(\theta T)^2}\right)$$

4. Inference for the discretely observed telegraph process

We present results on the estimation of λ when the telegraph process is observed at discrete times only $t_i = i\Delta_n$, $i = 0, 1, \ldots, n$ and we consider the asymptotics as $n \to \infty$, $\Delta_n \to 0$ and $n\Delta_n = T$ (i.e. high frequency)

Next picture is an example of a telegraph process observed discretely, i.e. the sample observations of X are $X(t_i) = X_i$



X_0
X_1
X_2
X_3
X_4
X_5
0
−5
−10
−15
X(t)
0
4
8
12
16
20
t

4. Inference for the discretely observed telegraph process

In the discrete case we do not observe all the velocity changes, so we can only have partial knowledge on λ.

The 2-dim process $(X(t), V(t))$ is Markovian but it is not reasonable to assume joint observation of the position and velocity of the particle.

We suppose that the velocity v is known. This hypothesis is not too restrictive and it is different from assuming that $V(t)$ is observable.

4. Inference for the discretely observed telegraph process

Consider once again the density of the distribution of $X(t)$

$$p(t, x; t_0, x_0) =$$

$$\frac{e^{-\lambda(t-t_0)}}{2v} \left\{ \lambda I_0 \left(\frac{\lambda}{v} \sqrt{u_t(x, x_0)} \right) + \frac{v\lambda(t-t_0) I_1 \left(\frac{\lambda}{v} \sqrt{u_t(x, x_0)} \right)}{\sqrt{u_t(x, x_0)}} \right\} \chi_{\{u_t(x,x_0)>0\}}$$

$$+ \frac{e^{-\lambda t}}{2} \delta(u_t(x, x_0))$$

The quantity

$$u_t(x, x_0) = v^2(t-t_0)^2 - (x-x_0)^2$$

takes a crucial role in the statistical analysis of the process when it is observed at discrete times. Indeed, if no Poisson event occurs from t_0 to t then $u_t(x, x_0) = 0$, otherwise it is positive.

4. Inference for the discretely observed telegraph process

The idea (De Gregorio, & I. ,2006) is to approximate the true (unknown) likelihood by the following quantity

$$L_n(\lambda) = \prod_{i=1}^{n} p(\Delta_n, X_i; t_0, X_{i-1})$$

where $p(t, x; t_0, x_0)$ is as in previous slide.

This means that we treat the increments of the process

$$\eta_i = X_i - X_{i-1} = X(i\Delta_n) - X((i-1)\Delta_n)$$

as n independent copies of $X(\Delta_n)$ with different initial value $X(t_0) = X_{i-1}$

4. Inference for the discretely observed telegraph process

Define the following pseudo-maximum likelihood estimator

$$\bar{\lambda}_n = \arg\max_{\lambda > 0} L_n(\lambda)$$

Theorem: The estimator $\bar{\lambda}_n$ is unique.

Proof is based on (tedious) manipulation of Bessel's functions.

4. Inference for the discretely observed telegraph process

Theorem: Under the condition $n\Delta_n = T$ as $\Delta_n \to 0$, the estimator $\bar{\lambda}_n$ converges to the maximum likelihood estimator in the case of complete observations, i.e. is such that

$$\bar{\lambda}_n \to \hat{\lambda}_\infty = \frac{N(T)}{T}\,.$$

This means that the pseudo-likelihood estimator shares the same asymptotic properties of the true limiting MLE

4. Inference for the discretely observed telegraph process

Consider the sample second moment of the observed increments $\eta_i = X_i - X_{i-1}$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} \eta_i^2$$

then the following least squares estimator can be considered

$$\tilde{\lambda}_n = \arg\min_{\lambda>0} \left\{ m_2 - \frac{v^2}{\lambda}\left(\Delta_n - \frac{1-e^{-2\lambda\Delta_n}}{2\lambda}\right)\right\}^2$$

In De Gregorio & I. (2006) numerical experiments on the approx. MLE and LS estimators ($\bar{\lambda}$ and $\tilde{\lambda}_n$) for small n and T fixed show good performance of both estimators.

4. Numerical results for the telegraph process

Next slide presents numerical results on the two estimators for different values of λ and sample size n. The average bias and mean square error (MSE) are presented along with support of the estimators

As it can be seen, for small sample size, the bias is an inverse function of the unknown parameter λ. This is expected because the trajectory is less informative.

4. Numerical results for the telegraph process

λ	Bias	$\sqrt{\text{MSE}(\lambda)}$	$\min \hat{\lambda}$	$\max \hat{\lambda}$	n
0.10	−0.002	0.018	0.04	0.20	50
	−0.001	0.016	0.05	0.16	100
	−0.000	0.014	0.05	0.15	1000
0.25	−0.011	0.041	0.13	0.47	50
	−0.003	0.031	0.16	0.41	100
	−0.000	0.023	0.16	0.35	1000
0.50	−0.062	0.092	0.26	0.78	50
	−0.011	0.059	0.32	0.85	100
	−0.000	0.033	0.37	0.63	1000
0.75	−0.151	0.175	0.36	1.01	50
	−0.031	0.091	0.47	1.18	100
	−0.000	0.043	0.60	0.92	1000
1.00	−0.264	0.283	0.45	1.23	50
	−0.064	0.128	0.62	1.53	100
	−0.001	0.051	0.81	1.22	1000
1.50	−0.546	0.558	0.58	1.48	50
	−0.162	0.227	0.90	2.16	100
	−0.001	0.066	1.27	1.77	1000
2.00	−0.874	0.882	0.75	1.65	50
	−0.298	0.357	1.11	2.67	100
	−0.000	0.083	1.68	2.33	1000

Pseudo MLE estimator $\hat{\lambda}$. $T = 500$, 10000 MC replications

4. Numerical results for the telegraph process

λ	Bias	$\sqrt{\text{MSE}(\lambda)}$	$\min \tilde{\lambda}$	$\max \tilde{\lambda}$	n
0.10	0.002	0.022	0.04	0.24	50
	0.001	0.018	0.05	0.18	100
	−0.000	0.016	0.05	0.16	1000
0.25	0.007	0.051	0.13	0.55	50
	0.003	0.037	0.14	0.41	100
	0.000	0.025	0.17	0.35	1000
0.50	0.018	0.106	0.25	1.13	50
	0.007	0.070	0.32	0.93	100
	0.000	0.037	0.35	0.65	1000
0.75	0.028	0.161	0.40	1.72	50
	0.012	0.105	0.45	1.32	100
	0.000	0.048	0.58	0.96	1000
1.00	0.040	0.219	0.53	2.42	50
	0.017	0.141	0.62	1.80	100
	0.001	0.057	0.77	1.25	1000
1.50	0.059	0.329	0.72	3.00	50
	0.028	0.218	0.92	2.78	100
	0.001	0.075	1.25	1.79	1000
2.00	0.080	0.412	1.05	3.00	50
	0.035	0.290	1.22	3.00	100
	0.003	0.095	1.66	2.39	1000

Least squares estimator $\tilde{\lambda}$. $T = 500$, 10000 MC replications

I. & Yoshida (2006) consider the same asymptotic scheme letting also $T \to \infty$ in order to have consistency and asymptotic normality of different estimators:

Theorem: let $\tilde{\lambda}_n$ be the moment type/LS estimator and suppose that $n\Delta_n \to \infty$, $\Delta_n \to 0$ as $n \to \infty$. Then, $\tilde{\lambda}_n$ is a consistent estimator of λ_0 and

$$\sqrt{n\Delta_n}(\tilde{\lambda}_n - \lambda_0) \xrightarrow{d} N\left(0, \frac{6}{5}\lambda_0\right)$$

as $n \to \infty$.

This estimator is not efficient.

Using moment expansion, is it possible to derive the following **explicit** moment-type estimator

$$\lambda_n^* = \frac{3}{2}\frac{1}{n\Delta_n}\sum_{i=1}^{n}\left\{1 - \frac{\eta_i^2}{v^2\Delta_n^2}\right\}$$

which is still consistent but not efficient.

Moreover, the additional assumption $n\Delta_n^3 \to 0$ is required in order to obtain asymptotic normality.

Finally, let

$$\check{\lambda}_n = \frac{1}{n\Delta_n} \sum_{i=1}^{n} \mathbf{1}_{\{|\eta_i| < v\Delta_n\}}$$

Then, the following estimator is also **efficient** under no additional assumptions

$$\hat{\lambda}_n = -\frac{1}{\Delta_n} \log\left(1 - \Delta_n \check{\lambda}_n\right)$$

i.e.

$$\sqrt{n\Delta_n}(\hat{\lambda}_n - \lambda_0) \xrightarrow{d} N(0, \lambda_0)$$

(Extensions of these results to random flights in R^2 observed at discrete times have been considered in De Gregorio (2007). In particular, the program of Ibragimov-Khasminskii was done for the large sample case.)

5. Geometric telegraph process

De Gregorio & I. (2006) considered also the geometric telegraph process defined as

$$S(t) = S(0)e^{\alpha t + \sigma X(t)}, \quad t > 0.$$

where $X(t)$ is the telegraph process with parameter λ, $\sigma > 0$ is called volatility and $\alpha = \mu - \frac{1}{2}\sigma^2$, with μ a known constant.

λ and σ are the parameters to be estimated from discrete observations $S_i = S(t_i)$ of the geometric telegraph process.

5. Geometric telegraph process

The *log-returns* Y_i are such that

$$Y_i = \log \frac{S_i}{S_{i-1}} = \alpha\Delta + \sigma\,\eta_i$$

where $S_i = S(t_i)$ and $\eta_i = X_i - X_{i-1}$. Then,

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i = \alpha\Delta + \frac{\sigma}{n} X_n$$

Hence

$$\mathsf{E}\bar{Y}_n = \alpha\Delta + \frac{\sigma}{n}\mathsf{E}X_n = \alpha\Delta = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta$$

Therefore,

$$\sigma^2 = 2\left(\mu - \frac{\mathsf{E}\bar{Y}_n}{\Delta}\right)$$

5. Geometric telegraph process: an estimator of σ^2

We propose the following unbiased estimator of σ^2

$$\hat{\sigma}_n^2 = 2\left(\mu - \frac{\bar{Y}_n}{\Delta}\right)$$

Please note that this estimator is not always well defined as there is no guarantee that

$$\mu \geq \frac{\bar{Y}_n}{\Delta}$$

for a single trajectory of the process.

5. Geometric telegraph process: an estimator of λ

The variance of the Y_i's is equal to

$$\mathrm{Var}\, Y_i = \sigma^2 \cdot \frac{v^2}{\lambda}\left(\Delta - \frac{1 - e^{-2\lambda\Delta}}{2\lambda}\right)$$

then the proposed estimator of λ is

$$\dot{\lambda}_n = \arg\min_{\lambda>0}\left(\bar{s}_Y^2 - \hat{\sigma}_n^2 \cdot \frac{v^2}{\lambda}\left(\Delta - \frac{1 - e^{-2\lambda\Delta}}{2\lambda}\right)\right)^2$$

where

$$\bar{s}_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2$$

5. Numerical results for the geometric telegraph process

Next slide presents numerical results on the two estimators $\hat{\sigma}_n^2$ and $\dot{\lambda}_n$ given the realized value of $\hat{\sigma}_n^2$.

Monte Carlo results over 10000 replications for different values of λ and sample size n. The average bias and mean square error (MSE) are presented along with support of the estimators

5. Numerical results for the geometric telegraph process

Please observe that since

$$\hat{\sigma}_n^2 = 2\left(\mu - \frac{\bar{Y}_n}{\Delta}\right) \quad \text{and} \quad \frac{\bar{Y}_n}{\Delta} = \alpha + \frac{\sigma}{n\Delta} X_n = \alpha + \frac{\sigma}{T} X_n$$

we have that $\hat{\sigma}_n^2$ is based on X_n only.

In our sampling scheme, we draw $n = 1000$ observations from the process, then we subsample the same path at $n = 500, 100, 50$ so X_n is always the same number and the estimator of σ^2 does not change in the following table.

5. Numerical results for the geometric telegraph process

λ	Bias	$\sqrt{\text{MSE}(\sigma)}$	min $\widehat{\sigma}_n$	max $\widehat{\sigma}_n$	% valid cases	n
0.10	−0.008	0.138	0.01	0.85	96	50, 100, 500, 1000
0.25	−0.007	0.094	0.02	0.74	99	50, 100, 500, 1000
0.50	−0.003	0.065	0.09	0.69	100	50, 100, 500, 1000
0.75	−0.002	0.053	0.22	0.66	100	50, 100, 500, 1000
1.00	−0.002	0.045	0.30	0.66	100	50, 100, 500, 1000
1.50	−0.002	0.037	0.33	0.64	100	50, 100, 500, 1000
2.00	−0.001	0.031	0.36	0.61	100	50, 100, 500, 1000

Empirical performance of the estimator $\widehat{\sigma}_n$ for different values of the parameter λ, different sample size and $\sigma = 0.5$. The time horizon $T = 500$, 10000 MC replications. '% valid cases' = percentage of valid cases, i.e. simulated paths such that the estimator of σ exists, i.e. $\mu \geq \bar{Y}_n/\Delta$.

5. Numerical results for the geometric telegraph process

λ	Bias	$\sqrt{\text{MSE}(\lambda)}$	min $\dot\lambda$	max $\dot\lambda$	% valid	n
0.10	0.018	0.107	0.00	1.01	96	50
	0.042	0.157	0.00	1.12	96	100
	0.634	1.189	0.00	6.48	96	1000
0.25	0.009	0.132	0.00	0.91	99	50
	0.006	0.161	0.00	0.95	99	100
	0.320	0.776	0.00	4.00	99	1000
0.50	0.022	0.185	0.00	1.64	100	50
	0.010	0.186	0.00	1.27	100	100
	0.139	0.635	0.00	3.33	100	1000
0.75	0.031	0.239	0.08	1.99	100	50
	0.014	0.215	0.00	1.90	100	100
	0.049	0.612	0.00	3.28	100	1000
1.00	0.045	0.304	0.31	2.68	100	50
	0.021	0.256	0.26	2.24	100	100
	0.012	0.608	0.00	3.69	100	1000
1.50	0.061	0.419	0.57	4.22	100	50
	0.028	0.330	0.59	3.22	100	100
	−0.014	0.580	0.00	3.97	100	1000
2.00	0.094	0.529	0.80	6.49	100	50
	0.038	0.402	0.93	4.16	100	100
	−0.002	0.536	0.00	4.08	100	1000

Empirical performance of the estimator $\dot\lambda_n$ given the estimate $\hat\sigma_n$. $T = 500$, 10000 MC replications, 'val' = % of valid cases, i.e. simulated paths such that $\mu \geq \bar{Y}_n/\Delta$.

Small announcement:

JSS (Journal of Statistical Software) is about to launch a call for paper for a special issue on “Computational Methods for S.D.E. and related models”

Any computer language is welcome!

JSS is an Open Source Journal of the American Statistical Association

`http://www.jstatsoft.org/`

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