Estimation of the Stationary Probability Measure of Semi-Markov Processes

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SAPS VI, March 21-23, 2007
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Plan

1. Preliminaries
2. Semi-Markov Processes
3. Stationary Probability Measure Estimation
4. Proofs
5. Related works

Preliminaries

Semi-Markov Processes is a generalization of Markov processes and Renewal Processes.
They include the major part of stochastic processes used in applications.

- Smith (1954), Lévy (1954) ; (Int. Conference on Mathematics in Amsterdam);
- "The mathematical theory is given by" R. Pyke (1961) ;
- Feller, Çinlar, Koroliuk, etc.

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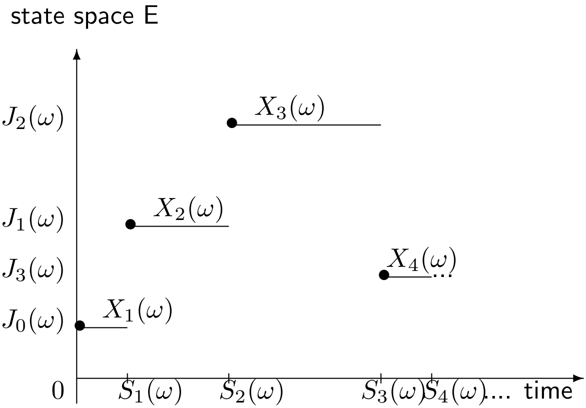
A trajectory



Related Processes

- $(J_n, S_n)_{n \in \mathbb{N}}$, **Markov renewal process** (MRP), with values in $E \times \mathbb{R}_+$;
- $(Z_t)_{t \in \mathbb{R}_+}$, **semi-Markov process** (SMP), with values in E;
- $(J_n)_{n \in \mathbb{N}}$, **embedded Markov chain** (EMC), with values in E.

$$N_t(\omega) = \sup\{n : S_n(\omega) \leq t\}$$

$$Z_t(\omega) = J_{N_t(\omega)}(\omega)$$

with $(E, \mathcal{E})$ a Borel-measurable space;

Markov Renewal Process

Markov renewal process $(J_n, S_n)_{n=0,1,2,...}$

$$\mathbb{P}(J_{n+1} \in B, S_{n+1} - S_n \leq t | J_0, ..., J_n; S_1, ..., S_n) \\ = \mathbb{P}(J_{n+1} \in B, S_{n+1} - S_n \leq t | J_n), \ a.s.$$

Semi-Markov kernel

$$Q(x, B, t) = \mathbb{P}(J_{n+1} \in B, S_{n+1} - S_n \leq t | J_n = x)$$

Transition probabilities of the EMC (J_n):

$$P(x, B) := Q(x, B, \infty) = \lim_{t \to \infty} Q(x, B, t).$$

Conditional distributions

$$F_{xy}(t) := \mathbb{P}(S_{n+1} - S_n \leq t | J_n = x, J_{n+1} = y).$$

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We have:

$$Q(x, dy, t) = P(x, dy) F_{xy}(t).$$

Particular case

Jump Markov process

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)]$$

Semi-Markov kernel

$$Q(x, B, t) = P(x, B)[1 - e^{-q(x)t}].$$

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Backward and Forward Recurrence Times

Let us consider for a fixed time $t > 0$, the r.v.s

$$U_t := t - S_{N(t)}, \quad \text{and} \quad V_t := S_{N(t)+1} - t,$$

and consider the backward and forward $(E \times \mathbb{R}_+, \mathcal{E} \times \mathcal{B}_+)$-valued processes $(Z_t, U_t, t \geq 0)$, and $(Z_t, V_t, t \geq 0)$ respectively.

Lemma

Both processes $(Z_t, U_t, t \geq 0)$, *and* $(Z_t, V_t, t \geq 0)$ *are Markov processes, with common stationary distribution*

$$\widetilde{\pi}(A \times \Gamma) := \frac{1}{\widetilde{m}} \int_A \nu(dx) \int_\Gamma [1 - F_x(u)] du, \qquad (1)$$

on $(E \times \mathbb{R}_+, \mathcal{E} \times \mathcal{B}_+)$.

Both processes are continuous from the right. The semigroup generated by these processes are strongly continuous.

Stationary probability of Z

Lemma

The marginal law $\pi(A) := \widetilde{\pi}(A \times \mathbb{R}_+)$, *on* $(E, \mathcal{E})$, *is the stationary probability measure of* $(Z(t), t \geq 0)$, *which is also the limit, as* $t \to \infty$, *of its transition function, that is*

$$\lim_{t\to\infty} \mathbb{P}(Z(t) \in A \mid Z(0) = x) = \pi(A) := \frac{1}{\widetilde{m}} \int_A \nu(dy) m(y),$$

for any $x \in E$.

Let us consider a measurable bounded function

$$v : E \to \mathbb{R}.$$

We intend to estimate functionals of the form

$$\widetilde{v} := \mathbb{P}_{\widetilde{\pi}} v = \int_E \pi(dx) v(x).$$

Estimators

We propose the following empirical estimator for $\widetilde{v}$:

$$\alpha_T := \frac{1}{T}\int_0^T v(Z(s))ds. \tag{2}$$

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$$\pi_T(A) := \frac{1}{T} \int_0^T \mathbf{1}_A(Z(s))ds. \quad (3)$$

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It is clear that estimators α_T and π_T are $\mathbb{P}_{\widetilde{\pi}}$-unbiased

$$\mathbb{P}_{\widetilde{\pi}}\alpha_T = \widetilde{v}, \quad \text{and} \quad \mathbb{P}_{\widetilde{\pi}}\pi_T(A) = \pi(A), \quad T > 0, A \in \mathcal{E}.$$

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We have also

$$\alpha_T \xrightarrow[T\to\infty]{\mathbb{P}_{(x,s)}} \widetilde{v}.$$

Asymptotic Properties of Estimators

- $\mathbb{P}_x$-stong consistency;
- $\mathbb{P}_x$-asymptotic normality;
- $\mathbb{P}_{\tilde{\pi}}$-strong invariance principle

$x \in E$.

Let us consider the family of processes $\alpha_T^\varepsilon, T \geq 0, \varepsilon > 0$,

$$\alpha_T^\varepsilon := \frac{1}{\varepsilon T}\int_0^T v(Z(s/\varepsilon^2))ds - \varepsilon^{-1}\tilde{v} = \frac{1}{\varepsilon T}\int_0^T v_0(Z(s/\varepsilon^2))ds, \quad (4)$$

where $v_0(x) := v(x) - \tilde{v}$.

Associated Markov Process to Z

Markov process $x^0(t), t \geq 0$.
Generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)],$$

where $q(x) = 1/m(x)$, $m(x) = \int_0^\infty \overline{F}(t)dt$.
Potential operator R_0

$$R_0 Q = Q R_0 = \Pi - I.$$

ASSUMPTIONS:

- A1 The semi-Markov process Z is uniformly ergodic.
- A2 The second moments of the sojourn times are uniformly bounded and

$$m_2(x) = \int_0^\infty t^2 F_x(dt) \leq M < +\infty,$$

and

$$\sup_x \int_T^\infty t^2 F_x(dt) \to 0, \quad T \to \infty.$$

The embedded Markov chain J is ergodic, with stationary distribution ν.

Weak Invariance Principle

Theorem

Under Assumptions A1-A2, the following weak convergence holds

$$\alpha_t^\varepsilon \Longrightarrow bW(t)/t, \quad \text{as } \varepsilon \to 0,$$

provided that $b^2 > 0$. *The variance coefficient* b^2 *is*

$$b^2 = b_0^2 + b_1,$$

where:

$$b_0^2 = 2\int_E \pi(dx)a_0(x), \quad a_0(x) = v_0(x)R_0v_0(x),$$

$$b_1 = \int_E \pi(dx)\mu(x)v_0^2(x), \quad \mu(x) = [m_2(x) - 2m^2(x)]/m(x).$$

Strong Invariance Principle

Let us put $\varepsilon := 1/\sqrt{n}$, $\alpha_n(t) := \alpha_t^\varepsilon$, and g be a function on $E \times \mathbb{R}_+$, defined by $g(x,s) = v_0(x)s$.

Lemma

The process (J_n, X_{n+1}) is an MRP, with semi-Markov kernel $\widetilde{Q}(x, dy \times ds) = P(x, dy)F_y(ds)$, where P is the transition kernel of the EMC (J_n), and $F_y(ds) = Q(y, E \times ds)$.
The stationary distribution of (J_n, X_{n+1}) is given by $\widetilde{\nu} := \nu F$, that is, $\widetilde{\nu}(dy \times ds) = \nu(dy)F_y(ds)$.

Iterates of transition kernel

$$M^{(1)}((x,s);B\times[0,t])=\widetilde{Q}(x,B,t-s)$$

$$M^{(n)}((x,s);B\times[0,t])=\int_{E\times\mathbb{R}_+}M^{(1)}((x,s);dy\times du)M^{(n)}((y,u);B\times[0,t])$$

Let us define also

$$M^{(n)}g(x,s)=\int_{E\times\mathbb{R}_+}M^{(n)}((x,s);dy\times du)g(y,u).$$

ADDITIONAL ASSUMPTIONS:

- A3 The MRP (J_n, X_{n+1}) is stationary, that is $\mathbb{P} \circ (J_0, X_1)^{-1} = \widetilde{\nu}$.
- A4 We have
$$\sum_{n \geq 1} \left\| M^{(n)} g \right\|_2 < \infty,$$
for the norm $\|f\|_2 := \left(\iint_{E \times \mathbb{R}_+} f^2 d\widetilde{\nu} \right)^{1/2}$ on $L^2(\widetilde{\nu})$.

Theorem

Let us assume that A1, A2, A3,A4, hold true. If moreover $\sigma^2 > 0$*, then the sequence* $\left\{ \frac{t\alpha_n(t)}{\sigma\sqrt{2\log\log n}}, n \geq 3 \right\}$*, viewed as a subset of* $C[0,1]$*, is* $\mathbb{P}_{\tilde{\pi}}$*-a.s. relative compact (in the uniform topology), and the set of its limit points is exactly* K*.*

$$K = \{x \in C : x \text{ is AC},\ x(0) = 0,\ \int_0^1 (\frac{d}{dt}x(t))^2 dt \leq 1\}.$$

$$\sigma^2 := \lim_{n\to\infty} \mathbb{P}\Big(\sum_{k=1}^{n} v_0(J_{k-1})X_k\Big)^2/n$$

Plan

Proof of Theorem 1 (Main steps)

STEP 1: COMPENSATING OPERATOR CONVERGENCE

Extended MRP

$$\alpha_n^\varepsilon := \alpha^\varepsilon(S_n^\varepsilon), \quad Z_n^\varepsilon := Z^\varepsilon(S_n^\varepsilon), \quad S_n^\varepsilon := \varepsilon^2 S_n$$

Compensating Operator (CO)

$$\mathbb{L}^\varepsilon\varphi(u,x,t) = \mathbb{E}\Big[\varphi(\alpha_1^\varepsilon, Z_1^\varepsilon, S_1^\varepsilon) - \varphi(u,x,t) \,|\, \alpha_0^\varepsilon = u, Z_0^\varepsilon = x, S_0^\varepsilon = t\Big]/m(x)$$

$$\mathbb{L}^\varepsilon\varphi(u,x) = \varepsilon^{-2} q(x)\Big[\int_0^\infty F_x(ds) A_{\varepsilon^2 s} P\varphi(u,x) - \varphi(u,x)\Big]$$

Troncated CO

$$\mathbb{L}_0^\varepsilon = \varepsilon^{-2} Q + \varepsilon^{-1} \mathbb{A}^\varepsilon(x) P + Q_2(x) P$$

Singular Perturbation Problem

$$\mathbb{L}_0^\varepsilon \varphi^\varepsilon(u,x) = \mathbb{L}\varphi(u) + \theta^\varepsilon(u,x)$$

Solution

$$\mathbb{L} = \Pi Q_2(x) \Pi + \Pi \mathbb{A}(x) P R_0 \mathbb{A}(x) P \Pi$$

where:

$$Q_2(x) = \frac{m_2(x)}{2m(x)} \mathbb{A}^2(x), \quad \mathbb{A}(x) = \varepsilon^{-1} v_0(x) \frac{d}{du}.$$

STEP 2: TIGHTNESS - Compact Containment Condition

$$\lim_{M\to\infty} \sup_{0<\varepsilon\leq\varepsilon_0} \mathbb{P}\Big(\sup_{0<t\leq T} |\alpha^{\varepsilon}(t)| \geq M \Big) = 0, \quad T>0$$

- Submartingale Condition

$$\eta^{\varepsilon}(t) := \varphi(\alpha^{\varepsilon}(t)) + C_{\varphi} t,$$

is an $(\mathcal{F}_t^{\varepsilon})$ nonnegative submartingale.
[This is an extension by Wentzell-Sviridenko to D space of Stroock-Varadhan theory on C space.]

Proof of Theorem 2 (Main steps)

Let us define

$$\xi(t) := \int_0^t v_0(Z(s))ds, \quad t \geq 0$$

$$\zeta_n := \sup\{|\xi(s) - \xi(S_n)| : S_n \leq s < S_{n+1}\}$$

$$b_n := \sigma(2n \log \log n)^{1/2}, \quad n \geq 3.$$

And the proof is as follows:

1. $\zeta_{[nt]}/b_n \xrightarrow[n\to\infty]{a.s.} 0$
2. the limit points of the process

$$\xi(S_{[nt]})/b_n, \quad n \geq 0,$$

   belong to K [Heyde and Scott (1973)];
3. From 1., 2. and [Serfozo (1975)], the proof is achieved.

Plan

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