Estimation and Forecast of non-constant Volatility[1]. Testing of Black-Scholes formula.

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1. Black-Scholes model for financial markets

The stock price $S(t)$ is modeled as a Geometric Brownian motion

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dB_t,\ 0 \le t \le T,$$

where
- B_t is a Brownian motion
- r is non-risky rate
- $v(t)$ is volatility (risky rate; positive and deterministic).

The averaged volatility $\frac{1}{T-t}\int_t^T v(s)ds$ involves in Black-Scholes formula for "prediction" $\mathrm{E}([S_T - K]^+ | S_t)$ of stock prices, where T and K are the maturity time and strike respectively.

2. Examples

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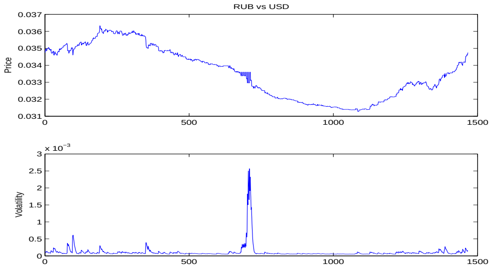
Exchange rate RUB vs USD. Fast oscillations with small amplitudes of the price provide a jumps of volatility



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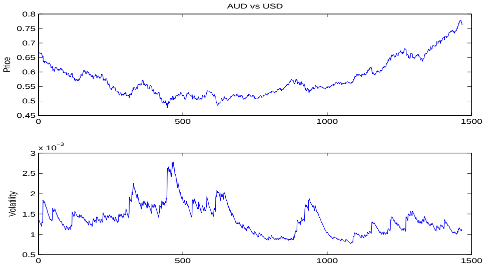
Exchange rate Australian vs USA dollars



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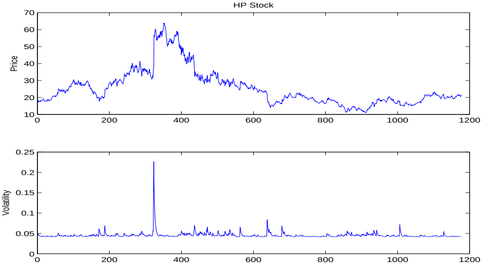
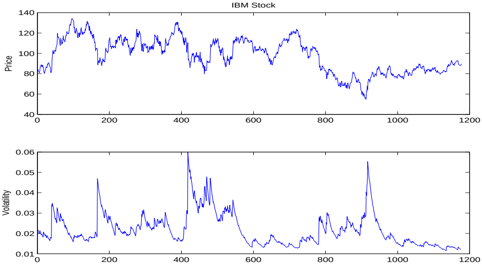
Hewlett-Packard stock



IBM stock


IBM Stock
Price
140
120
100
80
60
40
0
200
400
600
800
1000
1200
Volatility
0.06
0.05
0.04
0.03
0.02
0.01

6. Examples

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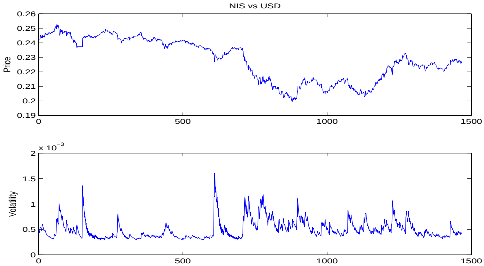
Exchange New Israeli Shekel vs USA dollar



7. Volatility as unknown function

Estimation of $v(t)$ via the sample $X_i = \frac{1}{\triangle}\log^2\left(\frac{S(t_i)}{S(t_{i-1})}\right)$, where $i = 0, 1, \dots, n, \quad t_i - t_{i-1} = \frac{T}{n}(=: \triangle)$.

Due to the Itô formula

$$\log\left(\frac{S(t_i)}{S(t_{i-1})}\right) = \int_{t_{i-1}}^{t_i} 0.5\big(2r - v(s)\big)ds + \int_{t_{i-1}}^{t_i} \sqrt{v(s)}dB_s,$$

and, for small $\triangle$,

$$X_i = \frac{1}{\triangle}\log^2\left(\frac{S(t_i)}{S(t_{i-1})}\right) \overset{\mathbb{L}^2}{=} \frac{1}{\triangle}\Big(\int_{t_{i-1}}^{t_i} \sqrt{v(s)}dB_s\Big)^2 + O\Big(\sqrt{\triangle}\Big)$$

8. $v(t)$ as smooth function

Then $v_{i-1} = \frac{1}{\triangle}\int_{t_{i-1}}^{t_i} v(s)ds$ is a reasonable approximation for $v(t)$, $t \in [t_{i-1}, t_i)$. Denote

$$\xi_i = \frac{1}{\sqrt{v_{i-1}\triangle}}\int_{t_{i-1}}^{t_i}\sqrt{v(s)}dB_s.$$

$(\xi_i)_{i\geq 1}$ is a sequence of i.i.d. $(0,1)$-Gaussian random variables.

We have

$$\underbrace{X_i}_{\text{measurement}} = \underbrace{v_{i-1}}_{\text{volatility}} + \underbrace{v_{i-1}(\xi_i^2 - 1) + O(\sqrt{\triangle})\xi_i}_{\text{“ noise“}\equiv\eta_i} + \underbrace{O(\triangle)}_{\text{small shift}}$$

9. GARCH TRACKING ALGORITHM

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GARCH = Generalized Autoregressive Conditional Heteroscedasticity

Following Bollerslev and Engle, (p, q)-GARCH tracking algorithm estimates v_i's with the help of recursive procedure (with appropriate initial conditions)

$$\widehat{v}_i = K + \sum_{j=1}^{p} g_j \widehat{v}_{i-j} + \sum_{m=1}^{q} a_m X_{(i+1)-m}.$$

Parameters $\boxed{K, g_1, \dots, g_p, a_1, \dots, a_q, p, q}$ are chosen by minimization of the innovation difference (here $n = \frac{T}{\triangle}$):

$$\frac{1}{n} \sum_{i=1}^{n} (X_i - \widehat{v}_{i-1})^2.$$

10. Alternative approach to GARCH algorithm

At first glance the GARCH filter corresponds is a version of the Wiener filter related to a stationary process with rational spectra.

We attempt to find an approach for creating GARCH-type filters based on ideas from

Nonparametric Statistic Inferences

It is known from Ibragimov - Khasminskii and Stone, there exists a kernel type estimate $\widehat{v}_i$ such that

$$\sup_{i \le n} \mathrm{E}(v_i - \widehat{v}_i)^2 \le O\big(n^{-2(1+k)/(2k+3)}\big)$$

provided that $v(t)$ possesses k derivative $v^{(j)}(t), j = 1, \dots, k$ and k-th derivative is Lipschitz continuous.

11. Alternative approach to GARCH algorithm

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Any GARTH filter requires an initial conditions implying some boundary layer, where the rate in n guaranteeing by nonparametric estimate might be lost. However, this boundary layer is not large: $i \leq O\big(n^{-1/(2k+3)} \log n\big)$, (Khasminskii and Liptser) and out of it: $i > O\big(n^{-1/(2k+3)} \log n\big)$,

$$\sup_{i \leq n} \mathrm{E}(v_i - \widehat{v}_i)^2 \leq O\big(n^{-2(1+k)/(2k+3)}\big).$$

This result can be obtained with GARCH of Kalman's type filter.

12. GARCH as Kalman filter

It is known from Khasminskii, Liptser that for k-times differentiable function $v(t)$ with the Lipschitz continuous elder derivative the Kalman filter providing the same rate rate in n: $\boxed{\sup_{i\le n}\mathrm{E}(v_i-\widehat{v}_i)^2\le O\big(n^{-2(1+k)/(2k+3)}\big)}$ has the following structure

$$
\begin{aligned}
\widehat{v}_i &= \widehat{v}_{i-1}+\frac{1}{n}\widehat{v}^{(1)}_{i-1}+\frac{q_0(\vartheta)}{n^{(2(k+1))/(2k+3)}}\big(X_i-\widehat{v}_{i-1}\big)\\
\widehat{v}^{(j)}_i &= \widehat{v}^{(j)}_{i-1}+\frac{1}{n}\widehat{v}^{(j+1)}_{i-1}+\frac{q_j(\vartheta)}{n^{(2(k+1)-j)/(2k+3)}}\big(X_i-\widehat{v}_{i-1}\big), j\le k-1\\
\widehat{v}^{(k)}_i &= \widehat{v}^{(k)}_{i-1}+\frac{q_k(\vartheta)}{n^{(k+2)/(2k+3)}}\big(X_i-\widehat{v}_{i-1}\big),
\end{aligned}
$$

for any $q_i(\vartheta)$'s guaranteeing the filter stability.

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A remarkable property of GARCH in the form of Kalman's filter is that all coefficients $q_i(\vartheta)$, $i = 0, 1, \ldots, k$ are parameterized by a scalar parameter ϑ

$$q_j(\vartheta) = U_{0j}\vartheta^{j+1/k+1}, \ i = 0, 1, \ldots, k,$$

$U_{0j}, \ j = 0, 1, \ldots, k$ are entries of a matrix U solving the algebraic Riccati equation: $\boxed{aU + Ua^* + B - UA^*AU = 0}$,

where matrices $\boxed{a, A, B}$ are related to "smooth assumptions" of $v(t)$ and a structure of the observation process.

14. Matrices a, A, B

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$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \; a = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

of corresponding sizes $1 \times (k+1)$; $(k+1) \times k+1$; $(k+1) \times 1$.

15. U_{0j}

For $k \leq 4$,

k	U_{00}	U_{01}	U_{02}	U_{03}	U_{04}
0	1	NA	NA	NA	NA
1	$\sqrt{2}$	1	NA	NA	NA
2	2	2	1	NA	NA
3	$\sqrt{4+\sqrt{8}}$	$2+\sqrt{2}$	$\sqrt{4+\sqrt{8}}$	1	NA
4	$1+\sqrt{5}$	$3+\sqrt{5}$	$3+\sqrt{5}$	$1+\sqrt{5}$	1

.

16. Global adaptation in ϑ

Measurements:

$$X_i = v_{i-1} + v_{i-1}(\xi_i^2 - 1) + O(\sqrt{\triangle})\xi_i + O(\triangle)$$

We have the filter with one free parameter $\boldsymbol{\vartheta}$:

$$\widehat{v}_i = \widehat{v}_{i-1} + \frac{1}{n}\widehat{v}^{(1)}_{i-1} + \frac{U_{00}\boldsymbol{\vartheta}^{\mathbf{1/k+1}}}{n^{2k+2/2k+3}}\big(X_i - \widehat{v}_{i-1}\big)$$

$$\widehat{v}^{(j)}_i = \widehat{v}^{(j)}_{i-1} + \frac{1}{n}\widehat{v}^{(j+1)}_{i-1} + \frac{U_{0j}\boldsymbol{\vartheta}^{\mathbf{(j+1)/k+1}}}{n^{(2(k+1)-j)/(2k+3)}}\big(X_i - \widehat{v}_{i-1}\big), \; j \le k-1$$

$$\widehat{v}^{(k)}_i = \widehat{v}^{(k)}_{i-1} + \frac{U_{0k}\boldsymbol{\vartheta}}{n^{(k+2)/(2k+3)}}\big(X_i - \widehat{v}_{i-1}\big).$$

17. Global Adaptation in ϑ

$$X_i = \underbrace{v_{i-1}}_{\text{volatility}} + \underbrace{v_{i-1}(\xi_i^2 - 1) + O(\sqrt{\triangle})\xi_i}_{\text{noise}} + \underbrace{O(\triangle)}_{\text{shift}}$$

$\boxed{\text{If } O(\triangle) \equiv 0,}$ the noise variance is bounded. Then, by Ibragimov - Khasminskii

$$\mathrm{E}(v_i - \widehat{v}_i)^2 \le O\big(n^{-2(1+k)/(2k+3)}\big),\ n \to \infty.$$

For $\boxed{O(\triangle) \not\equiv 0,}$ the same upper bound remains valid since

$$O(\triangle) \asymp \frac{1}{n}.$$

18. Adaptation procedure

It is naturale to choose (recall $\widehat{v}_{i-1} = \widehat{v}_{i-1}(\vartheta)$)

$$\vartheta^* = \operatorname*{argmin}_{\vartheta} \frac{1}{n}\sum_{i=1}^{n}(v_{i-1} - \widehat{v}_{i-1}(\vartheta))^2$$

Since $\boxed{v_{i-1}\text{'s}}$ are unknown(!), we are compelled to choose

$$\vartheta^{**} = \operatorname*{argmin}_{\vartheta} \frac{1}{n}\sum_{i=1}^{n}\Big(\underbrace{X_i}_{=v_{i-1}+\text{noise}+O(\triangle)} - \widehat{v}_{i-1}(\vartheta)\Big)^2.$$

Fortunately

$$\mathrm{P}\big(|\vartheta^{**} - \vartheta^*| > \delta\big) \asymp O\big(n^{-(4k+5)/(2k+3)}\big) = o\big(\underbrace{n^{-2(1+k)/(2k+3)}}_{\text{optimal rate in } n\to\infty}\big).$$

19. Filters controlled by a few parameters

$k = 0$ corresponds to Lipschitz continuous $v(t)$. This function can grow in t not faster than a linear function. In this case, we propose the estimator

$$\widehat{v}_i = \widehat{v}_{i-1} + \frac{\vartheta}{n^{2/3}}\big(X_i - \widehat{v}_{i-1}\big).$$

In the real world $v(t)$ is bounded function. As $v(t)$ is Lipschitz continuous and bounded, one may assume that for some positive number $\boldsymbol{a}$:

$$|\dot{v}(t) + \boldsymbol{a}v(t)| \leq \text{const.}$$

20. Filters controlled by a few parameters

The additional information provides a new filter

$$\widehat{v}_i = \widehat{v}_{i-1}\Big(1 - \frac{a}{n}\Big) + \frac{a\overline{X}}{n} + \frac{\vartheta}{n^{2/3}}\big(X_i - \widehat{v}_{i-1}\big),$$

where $\boxed{\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \approx \frac{1}{T}\int_0^T v(t)dt.}$ For a and $\overline{X}$, being sufficiently smaller than n, the rate in n is preserved:

$$\mathrm{E}(v_i - \widehat{v}_i)^2 \le O\big(n^{-2/3}\big).$$

The additional term $\boxed{-\widehat{v}_{i-1}\frac{a}{n} + \frac{\overline{X}}{n}}$ makes tracking error a little bit smaller owing to the boundary layer becomes shorter.

21. Filters controlled by a few parameters

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For $k \geq 1$, the filter is transformed to

$$\widehat{v}_i = \widehat{v}_{i-1} + \frac{1}{n}\widehat{v}^{(1)}_{i-1} + \frac{U_{00}\vartheta^{1/k+1}}{n^{2(k+1)/(2k+3)}}\big(X_i - \widehat{v}_{i-1}\big)$$

$$\widehat{v}^{(j)}_i = \widehat{v}^{(j)}_{i-1} + \frac{1}{n}\widehat{v}^{(j+1)}_{i-1} + \frac{U_{0j}\vartheta^{(j+1)/k+1}}{n^{(2(k+1)-j)/(2k+3)}}\big(X_i - \widehat{v}_{i-1}\big)$$

$$j = 1, \dots, k-1$$

$$\widehat{v}^{(k)}_i = \widehat{v}^{(k)}_{i-1}\Big(1 - \frac{a_1}{n}\Big) - \frac{1}{n}\Big(\sum_{\ell=2}^{k-1} a_\ell \widehat{v}^{(k-\ell)}_{i-1} + a_k \widehat{v}_{i-1} + a_k \overline{X}\Big)$$

$$+ \frac{U_{0k}\vartheta}{n^{(k+2)/(2k+3)}}\big(X_i - \widehat{v}_{i-1}\big).$$

22. Tuning procedure: Filter 1($k = 0$) and Filter 2(k=1)

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Filter 1

1. Set $\boldsymbol{a} = 0$, $\overline{X} = 0$ and find ϑ^*.
2. Find $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.
3. For fixed $\overline{X}$ and ϑ^*, find $\boldsymbol{a}^*$.
4. Local minimization in $(\vartheta, \overline{X}, \boldsymbol{a})$ in a vicinity of $(\vartheta^*, \overline{X}, \boldsymbol{a}^*)$.

Filter 2

The same procedure with $(\vartheta, \overline{X}, \boldsymbol{a_1}, \boldsymbol{a_2})$.

23. Comparison with classical GARCH

Table: Stocks of large computer manufacturers

Asset name	Filter type				
	Garch(1,1)	Garch(2,2)	Filter 0	Filter 1	Filter 2
DIS	4.309e-003	4.314e-003	4.329e-003	4.284e-003	4.285e-003
HPQ	3.741e-002	3.771e-002	3.615e-002	3.608e-002	3.608e-002
IBM	3.229e-003	3.228e-003	3.217e-003	3.199e-003	3.200e-003
INTC	1.232e-002	1.230e-002	1.235e-002	1.223e-002	1.232e-002
MAT	1.899e-002	1.879e-002	1.850e-002	1.817e-002	1.847e-002
SUN	5.144e-004	5.131e-004	5.143e-004	5.135e-004	5.138e-004
TOY	6.134e-003	6.128e-003	6.158e-003	6.098e-003	6.082e-003

A different quality of Filters 1, 2 and GARCH(1,1), (2,2) is provided by different tuning procedures. Filter 1 provides the best quality.

24. Calibration of Black-Scholes Model. Implied volatility.

When $v(t) \equiv v$ (classical Black-Scholes model)

$$\mathrm{E}\big([S_T - K]^+ | S_t\big) = S_t \Phi(h_t) - K e^{-r(T-t)} \Phi\big(h_t - \sqrt{v(T-t)}\big),$$

where Φ is the standard normal distribution function and

$$h_t = \frac{\log(S_t/K) + (r + \frac{1}{2}v)(T-t)}{\sqrt{v(T-t)}}.$$

Since a volatility $\boxed{v}$ is unknown, one can compute $\boxed{\text{implied volatility } v^{C_{T-t}}}$ which solves the equation

$$\mathrm{E}\big([S_T - K]^+ | S_t\big) =: C_t$$

with prescribed stock price C_t.

25. Calibration of Black-Scholes Model for time dependent volatility

For $v(t)$, the calibration can be done for $\boxed{\frac{1}{T-t}\int_t^T v(s)ds}$ since

$$\begin{aligned}&\mathrm{E}\big([S_T-K]^+|S_t\big)\\&=S_t\Phi(\widetilde{h}_t)-Ke^{-r(T-t)}\Phi\left(\widetilde{h}_t-\sqrt{\int_t^T v(s)ds}\right),\end{aligned}$$

where

$$\widetilde{h}_t=\frac{\log(S_t/K)+\int_t^T(r+\frac{1}{2}v(s))ds}{\sqrt{\int_t^T v(s)ds}}.$$

26. Volatility forecasting: historical + implied volatility

For $t = T - \triangle$, we compute the implied volatility $v^{C_\triangle}$.
For $m \le n-2$, we have estimates:
- $\widehat{v}_1, \dots, \widehat{v}_m$ - (estimates of historical volatility)
- $\widehat{v}_n := v^{C_\triangle}$ - (estimates of implied volatility)
We have typical smoothing problem:

$$\widehat{v}_1, \dots, \widehat{v}_m, \underbrace{v_{m+1}, \dots, v_{n-2}, v_{n-1}}_{\text{how to estimates these values?}}, \widehat{v}_n$$

A helpful tool here is a model of corresponding Kalman filter.

27. Example. IBM stock

Two filters have being used for tracking v_i, $1, \ldots, n$:

$$\widehat{v}_i = \left(1 - \frac{a}{n}\right) \widehat{v}_{i-1} + \frac{a\overline{X}}{n} + \text{“innov. difference”}$$

and

$$\widehat{v}_i = \widehat{v}_{i-1} + \frac{1}{n}\widehat{v}^{(1)}_{i-1} + \text{“innov. difference”},$$

$$\widehat{v}^{(1)}_i = \left(1 - \frac{a_1}{n}\right) \widehat{v}^{(1)}_{i-1} - \frac{a_2}{n}\widehat{v}_{i-1} + \frac{a_2\overline{X}}{n} + \text{“innov. difference”}$$

28. Smoother 1

- $n = 1200$
- $m = 1100$

Smoother 1.

$$\widetilde{v}_i = \left(1 - \frac{a}{n}\right)\widetilde{v}_{i-1} + \frac{a\overline{X}}{n} + u_i, \ \widetilde{v}_m = \widehat{v}_m, \ i > m,$$

where u_i, $i \geq m$ is a control action chosen such

$$\widetilde{v}_n = \widehat{v}_n \quad \text{and} \quad \sum_{i=m+1}^{n} u_i^2 \text{ is minimal.}$$

29. Simulation. Smoother 1 is not successful

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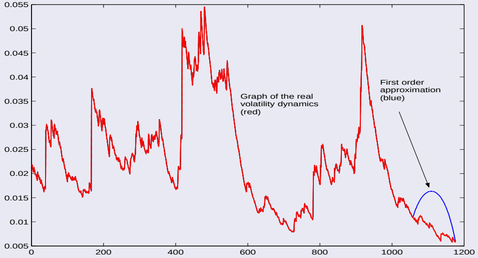

30. Smoother 2

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- $n = 1200$
- $m = 1100$

Smoother 2. For $i > m$

$$\widetilde{v}_i = \widetilde{v}_{i-1} + \frac{1}{n}\widetilde{v}^{(1)}_{i-1} + u_i$$

$$\widetilde{v}^{(1)}_i = \left(1 - \frac{a_1}{n}\right)\widetilde{v}^{(1)}_{i-1} - \frac{a_2}{n}\widetilde{v}_{i-1} + \frac{a_2\overline{X}}{n},$$

where $\widetilde{v}_m = \widehat{v}_m$ and $\widetilde{v}^{(1)}_m = \widehat{v}^{(1)}_m$ and a control action chosen such

$$\widetilde{v}_n = \widehat{v}_n \quad \text{and} \quad \sum_{i=m+1}^{n} u_i^2 \text{ is minimal.}$$

31. Simulation. Smoother 2 is successful