Statistique Asymptotique des Processus Stochastiques VI
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Continuous-Time Interpretation of Discrete-Time Market Data Revisited

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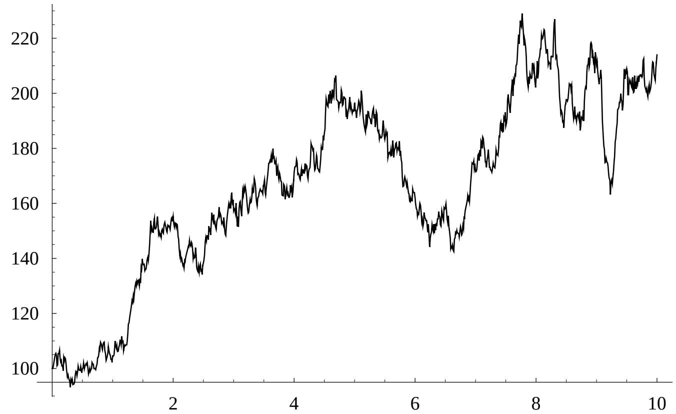
Market data?



$$X_t = M_t + A_t \quad (\mathcal{F}_t, P)\text{-semimartingale}$$

$$X_t = X_0 + \int_0^t \sigma(s, X_{0\leftrightarrow s})\, dW_s + \int_0^t \mu(s, X_{0\leftrightarrow s})\, ds$$

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\, dW_s + \int_0^t \mu(s, X_s)\, ds$$

$$\sigma(\cdot, \cdot) = ? \quad \mu(\cdot, \cdot) = ?$$

net return between t and $t+\Delta$: $\dfrac{X_{t+\Delta} - X_t}{X_t} = ?$ t = Δ, 2Δ, ...

$$\text{law of } \left\{ r_1 = \frac{X_\Delta - X_0}{X_0},\ r_2 = \frac{X_{2\Delta} - X_\Delta}{X_\Delta},\ \ldots,\ r_{k+1} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{X_{k\Delta}} \right\} = ?$$

relation to X_0 ?

Efficient Markets Theory

"efficiency" of the capital markets $\Longleftrightarrow$ the returns must be a "fair game"

attr. to E. Fama (1965), E. Fama (1970) (conditional expectations with respect to available information are the same as conditional expectations relative to market data)

Precursor: the random walk hypothesis

$$\{\log(X_\Delta),\ \log(X_{2\Delta}),\ \log(X_{3\Delta}),\ \dots\} = \text{random walk}$$

L. Bachelier (1900) [without the log=without the limited liability feature], M. G. Kendall (1953), C. Cranger & O. Morgenstern (1963)

Efficiency $\Longleftrightarrow$ all of the available information is fully reflected in the price

Can one reconcile the random walk hypothesis with concepts of "intrinsic" or "fundamental" value?

P. Samuelson (1965) "*Proof that properly anticipated prices fluctuate randomly*"

P. Samuelson (1965) "*Rational Theory of Warrant Pricing*"

P. Samuelson (1965) "*Proof that properly discounted present values of assets vibrate randomly*"

$$\text{NPV} \implies X_t = \sum_{i=1}^{\infty} \frac{1}{(1+\rho)^i} D_{t+i}$$

$$\xi_t := D_t / D_{t-1}$$

$$X_t = D_t \times \sum_{i=1}^{\infty} \frac{\xi_{t+1} \cdots \xi_{t+i}}{(1+\rho)^i}$$

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$$D_t = \xi_1 \xi_2 \dots \xi_t \ \text{(i.i.d.)}, \ \ \xi_t \perp\!\!\!\perp \mathcal{F}_{t-1}, \ \mathbb{E}[\xi_t] = \theta$$

$$X_t = D_t \times \sum_{i=1}^{\infty} \mathbb{E}\Big[\frac{\xi_{t+1} \cdots \xi_{t+i}}{(1+\rho)^i} \Big| \mathcal{F}_t\Big] = D_t \sum_{i=1}^{\infty} \frac{\theta^i}{(1+\rho)^i} = D_t \frac{\theta}{1+\rho-\theta}$$

$$\sim \ X_t = \frac{1}{1+\rho} \mathbb{E}[X_{t+1} + D_{t+1} \,|\, \mathcal{F}_t] \Longleftrightarrow X_{t+1} - X_t = \rho\, X_t - \theta\, D_t + X_t(\xi_{t+1} - \theta)$$

$$\Longleftrightarrow \ X_{t+1} - X_t = (\theta - 1)\, X_t + X_t(\xi_{t+1} - \theta) \ \Longleftrightarrow \ \frac{X_{t+1} - X_t}{X_t} = (\theta - 1) + (\xi_{t+1} - \theta)$$

$$\boxed{X_{t+1} = X_t\, \xi_{t+1}}$$

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Some Naïve Questions:

• If X_t "fully reflects" certain information about the return $\frac{X_{t+\Delta} - X_t}{X_t}$, can X_t and $\frac{X_{t+\Delta} - X_t}{X_t}$ be independent random variables?

• Can information disappear once it is taken into account?

• Does "predictability" in the returns mean an opportunity for extra profits? When such opportunities are exercised does the "predictability" go away?

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Empirical evidence for the validity of the random walk hypothesis:

M. G. Kendall (1953), C. Cranger & O. Morgenstern (1963),

E. Fama (1965), E. Fama (1970)

S. LeRoy (1989) "Efficient Capital Markets and Martingales"

• mean reversion, • fads

J. Campbell, A. Lo, A. C. MacKinlay (1997) "The Econometrics of Financial Markets"

• ratio test & $\psi_*(q)$ statistics

Are the returns $\{r_1, r_2, ..., r_{n\,q+1}\}$ independent? ($r_t = x_t - x_{t-1}$, $x_t := \log(X_t)$)

$$H_0: \ r_t = \mu + \epsilon_t \,, \quad \epsilon_t \in \mathcal{N}(0, \sigma^2)\,, \ \text{i.i.d.}$$

$$\text{VR}(q) := \frac{\text{Var}[r_t + r_{t-1} + \cdots + r_{t-q+1}]}{q\,\text{Var}[r_t]} = 1 + 2\sum_{k=1}^{q-1}\left(1 - \frac{k}{q}\right)\rho(k)$$

$$\hat{\mu}(q) := \frac{1}{n\,q}\,(r_{n\,q+1} - r_1)$$

$$\hat{\sigma}_a^2(q) := \frac{1}{n\,q-1}\sum_{k=1}^{n\,q}(x_{k+1} - x_k - \hat{\mu}(q))^2$$

$$\hat{\sigma}_b^2(q) := \frac{1}{m}\sum_{k=1}^{n\,q}(x_{k+1} - x_{k-q+1} - q\,\hat{\mu}(q))^2$$

$$m := q\,(n\,q - q + 1)\left(1 - \frac{q}{n\,q}\right)$$

Fact: $\hat{\sigma}_a^2(q)$ and $\hat{\sigma}_b^2(q)$ are asymptotically normal.

$$\hat{\text{VR}}(q) := \frac{\hat{\sigma}_a^2(q)}{\hat{\sigma}_b^2(q)}\,, \quad \hat{\text{VD}}(q) := \hat{\sigma}_b^2(q) - \hat{\sigma}_a^2(q)\ .$$

Hausman (1978): *The asymptotic variance of the difference between a consistent estimator* ($\hat{\sigma}_b^2(q)$) *and asymptotically efficient estimator* ($\hat{\sigma}_a^2(q)$) *equals the difference of the asymptotic variances.*

$$\sqrt{2\,n}\ \hat{\text{VD}}(2) \sim \mathcal{N}(0, 2\,\sigma^4)$$

Note: $2\,(\hat{\sigma}_a^2(2))^2$ is a consistent estimator of $2\,\sigma^4 \implies$

$$\frac{\sqrt{2\,n}\ \hat{\text{VD}}(2)}{\hat{\sigma}_a^2(2)\sqrt{2}} = \frac{\sqrt{2\,n}\ (\hat{\sigma}_b^2(2) - \hat{\sigma}_a^2(2))}{\hat{\sigma}_a^2(2)\sqrt{2}} = \sqrt{n}\left(\hat{\text{VR}}(2) - 1\right) \sim \mathcal{N}(0, 1)$$

There is a well known generalization for $q \geqslant 2$.

Campbell-Lo-MacKinlay test the following hypothesis:

$\boldsymbol{H_0^*}$: **1)** $\{\epsilon_t\}$ are uncorrelated; **2)** $\{\epsilon_t\}$ is ϕ-mixing with coefficients $\phi(m)$ of size $\frac{p}{2\,p-1}$ where $p \ni: \ \mathbb{E}[|\epsilon_t\,\epsilon_{t-\tau}|^{2\,(p+\delta)}] < \infty$ for some $\delta > 0$; **3)** $\lim_{nq\to\infty}(n\,q)^{-1}\sum_{t=1}^{nq}\mathbb{E}[{\epsilon_t}^2] = \sigma^2 < \infty$; **4)** $\mathbb{E}[\epsilon_t\,\epsilon_{t-j}\,\epsilon_t\,\epsilon_{t-k}] = 0$, $j \neq 0$, $k \neq 0$, $j \neq k$.

by using

$$\psi^*(q) := \frac{\sqrt{n \times q} \times (\overline{\mathrm{VR}}(q) - 1)}{\sqrt{\hat{\theta}(q)}} \sim \mathcal{N}(0, 1)$$

$$\hat{\theta}(q) := 4 \times \sum_{k=1}^{q-1} \left(1 - \frac{k}{q}\right)^2 \hat{\delta}_k(q)$$

$$\hat{\delta}_k(q) = \frac{n\, q \sum_{j=k+1}^{n\, q} \left((x_{j+1} - x_j - \hat{\mu})^2\, (x_{j-k+1} - x_{j-k} - \hat{\mu})^2\right)}{\left(\sum_{j=1}^{n\, q} (x_{j+1} - x_j - \hat{\mu})^2\right)^2}$$

$$\text{Test}: \ \psi^*(q) \in [-1.96, +1.96] \ ?$$

Based on this test Campbell-Lo-MacKinlay conclude that

"The average variance ratio with $q = 2$ is 0.96 for the 411 individual securities, implying that there is negative serial correlation on average. For all stocks, the average serial correlation is -4%, and -5% for the smallest 100 stocks. However, the serial correlation is both statistically and economically insignificant and provides little evidence against the random walk hypothesis. For example, the largest average $\psi^*(q)$ statistics over all stocks occurs for $q = 4$ and is -0.90 (with a cross sectional standard deviation of 1.19); the largest average $\psi^*(q)$ for the 100 smallest stocks is -1.67 (for $q = 2$ with a cross sectional standard deviation of 1.75). These results are consistent with French and Roll's (1986) finding that daily returns of individual securities are slightly negatively autocorrelated."

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But how powerful is the above test?

$$dX_t = dX_t - X_t\, dt, \ t \geqslant 0,$$

$$X_{\frac{1}{52}} - X_0, X_{\frac{2}{52}} - X_{\frac{1}{52}}, \ldots, X_{\frac{n}{52}} - X_{\frac{n-1}{52}}, \ldots,$$

$$X_{\frac{1}{365}} - X_0, X_{\frac{2}{365}} - X_{\frac{1}{365}}, \ldots, X_{\frac{n}{365}} - X_{\frac{n-1}{365}}, \ldots,$$

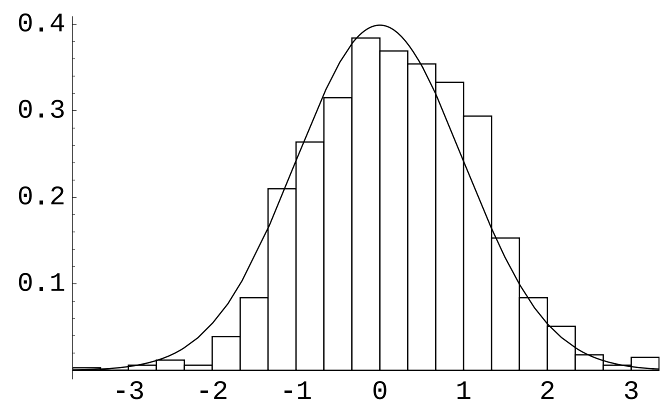
For the respective serial correlations we get

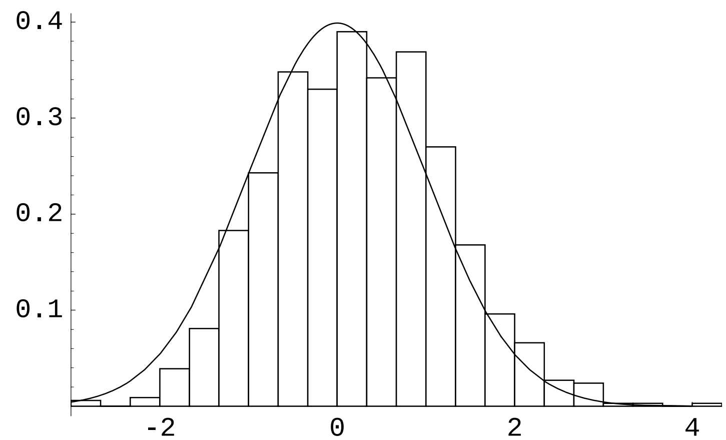
$$\frac{e^{-\frac{1}{52}}(1 - \mathrm{Cosh}(\frac{1}{52}))}{1 - e^{-\frac{1}{52}}} \approx -0.00952352 \ \text{ and } \ \frac{e^{-\frac{1}{365}}\left(1 - \mathrm{Cosh}\left(\frac{1}{365}\right)\right)}{1 - e^{-\frac{1}{365}}} = -0.00136799$$

and the respective variance ratios are

$$0.990476 \ \text{ and } \ 0.998632.$$

Moreover, the Monte-Carlo simulation of $\psi^*(q)$ for 1000 pseudo-random time series with $T = 33$ looks like (resp. for $q = 2$ and for $q = 4$)





In both cases $\psi^*(q) \in [-1.96, +1.96]$ in more than 95 % of the trials.

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Pro-gression vs. Re-gression type models:

$$X_{t+1} = X_t\,\xi_{t+1} \implies X_{t+k} = X_t\,\xi_{t+1} \ldots \xi_{t+k} \quad \text{(re-gression)}$$

future = function of the past + “noise”

present = function of the future + “noise”

$$X_t = \frac{1}{\xi_{t+1}}\,X_{t+1} \implies X_t = \frac{1}{\xi_{t+1} \ldots \xi_{t+k}}\,X_{t+k} \quad \text{(re-gression)}$$

$$X_0 = \frac{1}{\xi_1 \ldots \xi_{t+1}}\,X_{t+1},\;\; X_1 = \frac{1}{\xi_2 \ldots \xi_{t+1}}\,X_{t+1},\;\; X_t = \frac{1}{\xi_{t+1}}\,X_{t+1}$$

An example: a pro-gression type description of Brownian motion

$$Z_t := t \times \left(\xi + \int_t^1 s^{-1}\,dW_s\right),\;\; 0 \leqslant t \leqslant 1\,.$$

A re-gression-pro-gression type model for asset prices:

$$X_t = \underbrace{\int_0^t e^{-\sigma (W_s - W_t) - (\mu - \frac{1}{2}\sigma^2)(s-t)} C \, d\tau}_{A_t :=} + \underbrace{\int_t^\infty e^{-\sigma (W_s - W_t) - (\mu + \frac{1}{2}\sigma^2)(s-t)} C \, d\tau}_{P_t :=}$$

$$B_t := W_t - \sigma \int_0^t \left(1 + \frac{2 (C - P_s (\sigma^2 + \mu))}{{P_s}^2 \sigma^2}\right) ds$$

$$\frac{d X_t}{X_t} = \sigma \, d B_t + \left(\frac{2 C}{P_t} - \sigma^2 \times \left(1 - \frac{P_t}{X_t}\right) - \mu \right) dt \; , \; t \geqslant 0 \, .$$

$$d P_t = \sigma P_t \, d B_t + (C - \mu P_t) \, dt \; , \; t \geqslant 0 \, ,$$

$$d X_t = \sigma X_t * d W_t + \mu X_t \, dt$$

$$X_0 = \int_0^\infty e^{-\sigma W_s - (\mu + \frac{1}{2}\sigma^2) s} C \, ds$$