Statistical Problems on the number of delays in SDE with delays

T. MOURID Université Abou Bekr - Tlemcen - Algerie

Diffusion processes $(X^{\varepsilon}(t), t \in [0,T])$ for $\varepsilon > 0$:

$$dX^{\varepsilon}(t) = \left(\int_0^{\delta} X^{\varepsilon}(t-s)\, \mu(ds) \right) dt + \varepsilon\, dW(t) \quad , t \in [0,T]$$

μ signed measure with support $[0,\delta]$, $\delta > 0$, $\|\mu\| \leq L$, $L > 0$

$(W(t), t \in [0,T])$ Wiener Process

Initial condition : $\left[\begin{array}{l} X^{\varepsilon}(t) = x_0(t) \text{ if } t \in [-\delta, 0]. \\ \text{with a given } x_0 . \end{array} \right.$

Processes with memory of length δ

Deterministic model.

$$\frac{d x(t)}{dt} = \int_0^{\delta} x(t-s)\, \mu(ds) \quad , t \in [0,T]$$

$$x(t) = x_0(t) \quad , t \in [-\delta, 0].$$

How to recover μ and support of μ.
when $\varepsilon \to 0$? (small diffusion)

When
$$\mu = \sum_{i=1}^{k} a_i \delta_{b_i} \quad , a_i > 0, \quad 0 < b_1 < \dots < b_k$$

we have

$$d X^\varepsilon(t) = \Big(\sum_{i=1}^{k} a_i X(t-b_i)\Big) dt + \varepsilon\, dW(t)$$

$$X^\varepsilon(s) = x_0 > 0 \quad , -2b_k \le s \le 0 .$$

the parameter $\theta = (a_1, \dots, a_k, b_1, \dots, b_k)' \in \mathbb{R}^{2k}$, k known

- In Bosq - Kutoyants - Mourid (92)

• $(P_\theta^\varepsilon, \theta \in \Theta) \to$ Condition LAN (Hajek - Ibragimov - Khasminski)

• MLE and BE. are consistent
asymptotic normality
efficient (Hajek).

As $\varepsilon \to 0$:
• $\hat{\theta}_\varepsilon \longrightarrow \theta_0$ (true value)
• $\varepsilon^{-1}(\hat{\theta}_\varepsilon - \theta_0) \Longrightarrow \mathcal{N}$ gaussian r.v.
• bound in Hajek's inequality is reached by $\hat{\theta}_\varepsilon$.

- In Kutoyants - Mourid (94)

estimation of $\quad f(t) = \int_0^\delta x(t-s)\,\mu(ds)$

by the kernel estimator:

$$\hat{f}_\varepsilon(t) = \frac{1}{\psi_\varepsilon} \int_0^T K\Big(\frac{s-t}{\psi_\varepsilon}\Big) dX^\varepsilon(s)$$

$\psi_\varepsilon \to 0$, $\varepsilon^2 \psi_\varepsilon^{-1} \to 0$, conditions on K we have:
uniform consistence, asymptotic normality for $\hat{f}_\varepsilon(t)$.

Minimum distance estimator for $\theta = (a_1, \dots, a_k, b_1, \dots, b_k)' \in \mathbb{R}^{2k}$

Define
$$\theta^*_\varepsilon = \underset{\theta \in \Theta}{\arg\min} \int_{a_\varepsilon}^{b_\varepsilon} \Big(\hat{f}_\varepsilon(t) - \sum_{i=1}^{k} a_i x(t-b_i) \Big)^2 \nu(dt)$$

$$\begin{cases} a_\varepsilon \to 0 \ , \ b_\varepsilon \to T \ , \ a_\varepsilon \psi_\varepsilon^{-1} \to +\infty \ , \ (T-b_\varepsilon)\psi_\varepsilon^{-1} \to +\infty \\ \varepsilon^2 \psi_\varepsilon^{-1} \to 0 \ , \ \psi_\varepsilon \to 0. \end{cases}$$

ν measure on $[0,T]$.

Define for $\beta > 0$:
$$g_\varepsilon(\beta) = \inf_{\theta_0 \in K} \inf_{|\theta-\theta_0| > \beta} \int_{a_\varepsilon}^{b_\varepsilon} \big(S(t,\theta,x_0) - S(t,\theta_0,x_{\theta_0}) \big)^2 \nu(dt)$$
where $S(t,\theta,x_0) = \sum_{i=1}^{k} a_i x(t-b_i)$

K compact $K \subset \Theta$

Results

- If $g_\varepsilon(\beta) > 0$ then for $\forall \varepsilon < \varepsilon_1$:
$$\sup_{\theta_0 \in K} P^\varepsilon_{\theta_0}\big(\| \theta^*_\varepsilon - \theta_0 \| > \beta \big) \leq 3 \exp\Big(-c \frac{g_\varepsilon(\beta)}{\varepsilon^{4/3}} \Big)$$

- If ν is discrete measure
$$\varepsilon^{-2/3} \big(\theta^*_\varepsilon - \theta_0 \big) \Rightarrow \xi. \text{ gaussian.}$$
where
$$\xi = I^{-1}(\theta_0) \int_0^T \Big[\Big(\int u K(u)\, du \Big) \Big(\int_0^\delta x'_{\theta_0}(t-s) \mu(ds) \Big) + Y^\circ(t) \Big] \dot{S}(t,\theta_0,x_0)\, \nu(dt)$$
$$Y^\circ(t) \sim \mathcal{N}\Big(0, \int K^2\Big) \ , \ E(Y^\circ(t) Y^\circ(s)) = 0 \text{ if } s \neq t.$$
$$I(\theta) = \big(\langle \dot{S}(t,\theta,x_0), \dot{S}(t,\theta,x_0)^T \rangle_{L^2(\nu)} \big)$$

Estimation of the support of μ.

$$\operatorname{Supp} \mu = [0, \delta] \quad , \delta > 0$$

$$\delta \in]0, \Delta[\quad , \Delta > 0.$$

Define

$$\delta_\varepsilon^* = \underset{0<\delta<\Delta}{\arg\min} \int_{a_\varepsilon}^{b_\varepsilon} \left(\hat{f}_\varepsilon(t) - \int_0^\delta x(t-s)\,\mu(ds) \right) \nu(dt).$$

Result.

- Under condition of identifiability, for $\forall \varepsilon < \varepsilon_1$ then $\forall \beta > 0$

$$\sup_{\delta_0 \in K} P_{\delta_0}^\varepsilon \left(|\delta_\varepsilon^* - \delta_0| > \beta \right) \le C_1 \exp\left(- C_2 \frac{h_\varepsilon(\beta)}{\varepsilon^{4/3}} \right)$$

where $h_\varepsilon(\beta) = \inf_{\delta_0 \in K} \inf_{|\delta - \delta_0| > \beta} \int_{a_\varepsilon}^{b_\varepsilon} (S(t,\delta,x_\delta) - S(t,\delta_0,x_{\delta_0}))^2 \nu(dt)$

- If ν is discrete

$$\varepsilon^{-2/3} (\theta_\varepsilon^* - \delta_0) \Rightarrow Y \quad \text{gaussian.}$$

-

$$Y = I^{-1}(\delta_0) \int_0^T \left[\left(\int u K(u) du\right) \left(\int_0^{\delta_0} x'(t-s)\,\mu(ds) \right) + Y^0(t) \right] \dot{S}(t,\delta_0,x_{\delta_0})\,\nu(dt)$$

$Y^0(t) \sim N\left(0, \int K^2\right)$, $E(Y^0(s)Y^0(t)) = 0 \ \forall s \neq t.$

$$I(\delta_0) = \int_0^T (\dot{S}(t,\delta_0,x_{\delta_0}))^2 \nu(dt)$$

Selection of the number of delays

we observe.

$$dX^\varepsilon(t) = \left(\sum_{i=1}^{k} a_i X(t-b_i)\right)dt + \varepsilon\, dW(t)$$

$+$ CI.

we suppose a_i, b_i known. $\quad \begin{array}{|l} a_i = f(i) \\ b_i = g(i) \end{array}$ f, g known

How to select the true value of k ?

the parameter $k \in \Theta = \{1, 2, .., K\}$, $K > 0$.

Maximum likelihood method.

P_k^ε probability law of $(X^\varepsilon(t), t \in [0,T])$.

the likelihood statistic $\hat{k}_\varepsilon$ is defined by

$$\frac{dP^\varepsilon_{\hat{k}_\varepsilon}}{dP^\varepsilon_{k_0}}(X^\varepsilon) = \max_{k \in \Theta} \frac{dP^\varepsilon_k}{dP^\varepsilon_{k_0}}(X^\varepsilon)$$

where k_0 is an arbitrary value in Θ.

Result. There exists C_1 et C_2 positife constants such that

$$\max_{k_0 \in \Theta} P^\varepsilon_{k_0}\left(\hat{k}_\varepsilon \neq k_0\right) \leq C_1 \exp\left(-\frac{C_2}{\varepsilon^2}\right)$$

(The same result holds for the Bayesian Method.)

So the probability to select the true model $> 1 - C_1 e^{-\frac{C_2}{\varepsilon^2}}$.

Minimum distance method.

Set $$S(t,k,x) = \sum_{i=1}^{k} a_i\, x(t-b_i)$$

$$\hat{f}_\varepsilon(t) = \frac{1}{\varphi_\varepsilon}\int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) dX^\varepsilon(t).$$

Define

$$k^*_\varepsilon = \arg\min_{k\in\Theta} \int_{a_\varepsilon}^{b_\varepsilon}\left(\hat{f}_\varepsilon(t) - \sum_{i=1}^{k} a_i x(t-b_i)\right)^2 \nu(dt).$$

Set $$\ell_\varepsilon(u) = \min_{k_0}\ \min_{|k-k_0|\geq u} \int_{a_\varepsilon}^{b_\varepsilon} \left(S(t,k,x_k) - S(t,k_0,x_{k_0})\right)^2 \nu(dt).$$

where $u \in \mathbb{N}^*$.

Result: If $\ell_\varepsilon(u) > 0$, then $\exists\, C > 0$ such that

$$\max_{k_0\in\Theta} P^\varepsilon_{k_0}\left(k^*_\varepsilon \neq k_0\right) \leq C \exp\left(-c\,\frac{\ell_\varepsilon(u)}{\varepsilon^{4/3}}\right)$$

Probability to select the true model $> 1 - C\exp\left(-c\,\frac{\ell_\varepsilon(u)}{\varepsilon^{4/3}}\right)$.

Misspecified model.

The observations satisfy the model

$$(1) \qquad dX^\varepsilon(t) = \left(\int_0^\delta X^\varepsilon(t-s)\mu(ds)\right)dt + \varepsilon\, dW(t). \qquad t\in[0,T]$$

$+$ CI.

but the statistician considers the model:

$$(2) \qquad dX^\varepsilon(t) = \left(\sum_{i=1}^{k} a_i X^\varepsilon(t-s_i)\right)dt + \varepsilon\, dW(t).$$

the parameter is $\theta = (a_1, \dots, a_k, b_1, \dots, b_k)$ k known

and construct MLE, BE. of θ.

Pb. What are the asymptotic properties of MLE in this situation?

(Book of Kutoyants (97).)

the observer does not know the true drift (or the drift is nonparametric) and proposes more simple or tractable drift ((a parametric drift).

Set

$$G(\theta) = \int_0^T \left(\int_0^\delta x(t-s)\mu(ds) - \sum_{i=1}^{k} a_i x(t-s)\right)^2 dt.$$

x deterministic sol. associated to (2)

In fact $G(\theta) = G(\theta, \mu, x)$.

Set $\theta^* = \underset{\theta\in\Theta}{\operatorname{Arg\,min}}\ G(\theta)$

Result.

If. G has a unique minimum at θ^*

$$\ddot{G}(\theta^*, \mu, x) > 0$$

$$\forall \alpha > 0, \quad \inf_{|\theta - \theta^*| > \alpha} |G(\theta, \mu, x) - G(\theta^*, \mu, x)| > 0$$

then. $\forall \varepsilon < \varepsilon_1$ et $\forall \rho > 0$

$$P_\theta^\varepsilon \left(\|\hat{\theta}_\varepsilon - \theta^*\| > \rho \right) \leq C_1 \exp\left(- \frac{C_2}{\varepsilon} \right)$$

$$C_1, C_2 > 0.$$

$\hat{\theta}_\varepsilon \to \theta^*$ the minimizer of $G(\theta)$

- **Some questions**
  - limit law for $\hat{\theta}_\varepsilon$!
  - expansions for $\hat{\theta}_\varepsilon$.!
  - misspecified model when the parameter is k!

References