Statistical Properties of Affine Stochastic Delay Differential Equations

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Outline

Stochastic ordinary differential equations (SODEs)

$$dX(t) = \mu_\theta(X(t))\,dt + \sigma_\theta(X(t))\,dB(t), \quad t \in [0, T],$$
$$X(0) = \xi, \quad \text{independent of } B(\cdot),\ \theta \in \Theta \subseteq \mathbb{R}^k,\ \Theta \text{ open.}$$

The solution is a Markov process

Statistical problem:

- Estimate θ, based on continuous time observation $X^T = (X(t),\ t \in [0, T])$.
- Study asymptotic properties for $T \to \infty$.

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Assumptions:

- $\sigma_\theta(x) = \sigma(x)$, independent of θ,
- $(X(t),\ t \geq 0)$ ergodic,
- ξ has the invariant distribution.

Kutoyants (2004), [12]

Maximum likelihood method

$$\ell_T := \log \frac{d\,\mathbb{P}_\theta^T}{d\,\mathbb{P}_{\theta_0}^T}$$
$$= \int_0^T \frac{\mu_\theta(X(s)) - \mu_{\theta_0}(X(s))}{\sigma^2(X(s))}\,dX(s) - \frac{1}{2}\int_0^T \frac{(\mu_\theta(X(s)) - \mu_{\theta_0}(X(s)))^2}{\sigma^2(X(s))}\,ds,$$

$$\hat{\theta}_T = \arg\max_{\theta\in\Theta} \ell_T(\theta, (X(t), t \in [0, T])).$$

$\mathbb{P}_\theta^T$ ... distribution of $(X(t), t \le T)$ on $C([0, T])$.

Method of localization

$$\eta = \theta + \phi_T(\theta) u,$$

where $u \in \mathbb{R}^k, \phi_T(\theta)$ $k \times k$-matrix, Ibragimov, Khasminskii (1981), [9].

$$\phi_T(\theta) \longrightarrow 0, \quad \text{for } T \to \infty. \qquad \text{(normalizing matrix)}$$

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where $u \in \mathbb{R}^k, \phi_T(\theta)$ $k \times k$-matrix.

Under some conditions of regularity:

The family $(\mathbb{P}_\theta^T,\ \theta \in \Theta)$ is locally asymptotically normal (LAN) at every $\theta \in \Theta$. It means

$$Z_{\theta,T}(u) := \log \frac{d\,\mathbb{P}^T_{\theta+\phi_T(\theta)u}}{d\,\mathbb{P}^T_\theta} \xrightarrow{T\to\infty} \langle u, \Delta(\theta)\rangle - \frac{1}{2}\langle I(\theta)u, u\rangle\,,$$

where

$$\Delta(\theta) \sim N(0, I(\theta)) \quad \text{and} \quad \phi_T(\theta) = T^{-1/2} J_k,$$

$$I(\theta) := \mathbb{E}_\theta \left(\frac{\dot{\mu}_\theta(\xi)\dot{\mu}^*_\theta(\xi)}{\sigma(\xi)} \right).$$

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where $u \in \mathbb{R}^k, \phi_T(\theta)$ $k \times k$-matrix.

It is well known, that in the non regular case other limits of the log-likelihood functions are possible, e.g., LAMN, LAQ cases and even

$$Z_\theta(u) = \exp\left\{ C_\theta B^H(u) - \frac{C_\theta^2}{2}|u|^{2H} \right\},$$

where $B^H(\cdot)$ is the fractional Brownian motion with Hurst parameter $H \in [1/2, 1]$, C_θ is constant.

The affine Markovian case (Ornstein-Uhlenbeck case)

$$dX(t) = \theta X(t)\,dt + dB(t), \quad t \in [0,T],$$
$$X(0) = \xi, \quad \text{independent of } B(\cdot).$$

$$\Longrightarrow X(t) = \xi\, e^{\theta t} + \int_0^t e^{\theta(t-s)}\,dB(s), \quad t \geq 0,$$

$$\ell_T = \log L_T = \log \frac{d\,\mathbb{P}_\theta^T}{d\,\mathbb{P}_0^T} = \theta \int_0^T X(t)\,dX(t) - \frac{\theta^2}{2}\int_0^T X^2(t)\,dt,$$

$$\hat{\theta} = \underbrace{\left(\int_0^T X^2(t)\,dt\right)^{-1}}_{I_T} \underbrace{\int_0^T X(t)\,dX(t)}_{V_T} = I_T^{-1} V_T.$$

The affine Markovian case (Ornstein-Uhlenbeck case)

$$\hat{\theta}(T) - \theta = \left(\int_0^T X^2(t)dt \right)^{-1} \int_0^T X(t)\, dB(t) = I_T^{-1} V_T^0.$$

Asymptotic properties of $\hat{\theta}(T)$ for $T \to \infty$.

Three different cases:

$\theta < 0:$ LAN,

$\theta > 0:$ LAMN,

$\theta = 0:$ LAQ.

The Markov property is paid by the assumption that the coefficient $\mu(X(t))$ and $\sigma(X(t))$ depend on the present state $X(t)$ of X only, i.e. that X has no memory.

In many cases there is some time between reason and effect of a phenomenon

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Biology and Population dynamics:

- time to maturity
- time to educate
- incubation period

Economics: time between

- taking a decision and occurrence of the feedback
- time to build, to transport, to store

The are different attempts to include memory effects into modeling real phenomena:

- CARMA process (Brockwell, 2001, [2])
- continuous time GARCH processes (Brockwell, Chadraa, Lindner, 2006, [1])

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Stochastic delay differential equations (SDDEs)

$$dX(t) = \mu_\theta(X_t)\,dt + \sigma_\theta(X_t)\,dB(t), \quad t \geq 0, \tag{1}$$
$$X_0 = (\xi(s), s \in [-r,0]), \tag{1'}$$

- X_0 - initial process, independent of B,
- $X_t = (X(t+s), s \in [-r,0])$: "segment of $X(\cdot)$ at time t",
- $r \geq 0$ length of the memory,
- $\mu_\theta(x), \sigma_\theta(x) \,:\, [0,\infty) \times C([-r,0]) \to \mathrm{IR}^1$, $\theta \in \Theta \subseteq \mathrm{IR}^k$.

The solutions of (1) and (1') are not Markov processes.
No explicit solutions, even in simple cases.

- Numerics of SDDEs was treated by Buckwar et al., [3], [4].
- The case of infinitely long memory ($r = \infty$) has some particular aspects, see e.g. Riedle, [16].

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Deterministic DDEs

$$\frac{dx(t)}{dt} = f(t, x_t), \quad t \geq 0,$$
$$x_0 = \xi \in C([-r, 0]).$$

- Widely developed theory: see e.g. Hale, Lunel, [8].

Stochastic DDEs

- Mao (1997, 2007), [13],
- Mohammed (1984), [14].

Affine stochastic delay differential equations (ASDDEs)

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)dt + dB(t), \quad t \geq 0, \tag{2}$$

$$X(u) = \xi(u), \quad u \in [-r, 0]. \tag{2'}$$

$a(\cdot)$ finite signed measure.
Autoregressive schema with continuous time.

Example (Ornstein-Uhlenbeck case)

$$\begin{aligned} a(ds) &= a\mathbf{1}_{\{0\}}(ds), \\ dX(t) &= aX(t)\,dt + dB(t), \\ X(0) &= \xi. \end{aligned}$$

The solution of the affine equation

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)\,dt + dB(t), \quad t \geq 0, \tag{2}$$

$$X(u) = \xi(u), \quad u \in [-r,0], \tag{2'}$$

admits the representation

$$X(t) = \xi(0)x_0(t) + \int_{-r}^{0}\int_{u}^{0} \xi(s)x_0(t+u-s)\,ds\,a(du) + \int_{0}^{t} x_0(t-s)\,dB(s).$$

The solution of the affine equation

$$dX(t) = \int_{-r}^{0} X(t+s)a(\,ds)\,dt + dB(t), \quad t \geq 0, \tag{2}$$

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$x_0(\cdot)$ denotes the so called fundamental solution,

$$\dot{x}_0(t) = \int_{-r}^{0} x_0(t+s)a(\,ds),$$

$$x_0(s) = \mathbf{1}_{\{0\}}(u), \quad u \in [-r,0].$$

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$$X(t) = \xi(0)x_0(t) + \int_{-r}^{0}\int_{u}^{0} \xi(s)x_0(t+u-s)\,ds\,a(du) + \int_{0}^{t} x_0(t-s)\,dB(s).$$

Example (Ornstein-Uhlenbeck case)

$$x_0(t) = \exp[at],$$

$$X(t) = \xi \exp[at] + \int_{0}^{t} \exp[a(t-s)]\,dB(s).$$

The fundamental solution $x_0(t)$:

$$\dot{x}_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$

$$x_0(t) = \mathbf{1}_{\{0\}}(t), \quad t \in [-r, 0].$$

The fundamental solution $x_0(\cdot)$

- has no explicit expression,
- may be oscillating instead of being monotone.

The fundamental solution $x_0(t)$:

$$\dot{x}_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$
$$x_0(t) = \mathbf{1}_{\{0\}}(t), \quad t \in [-r, 0].$$

Laplace-transform

$$\hat{x}_0(\lambda) = \int_{0}^{\infty} \mathrm{e}^{-\lambda s} x_0(s)\, ds, \quad Re\,\lambda > \|a\|,$$
$$\hat{x}_0(\lambda) = \left(\lambda - \int_{-r}^{0} \mathrm{e}^{\lambda s} a(ds)\right)^{-1}, \quad Re\,\lambda > \|a\|.$$

The fundamental solution $x_0(t)$:

$$\dot{x}_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$
$$x_0(t) = \mathbf{1}_{\{0\}}(t), \quad t \in [-r,0].$$

Characteristic function

The denominator

$$h(\lambda) = \lambda - \int_{-r}^{0} e^{\lambda s} a(ds), \quad \lambda \in \mathbb{C},$$

is called the characteristic function of the deterministic differential equation.

$h(\cdot)$ is holomorphic on $\mathbb{C}$.

The fundamental solution $x_0(t)$:

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$$h(\lambda) = \lambda - \int_{-r}^{0} \mathrm{e}^{\lambda s}\mathrm{a}(\mathrm{ds}), \quad \lambda \in \mathbb{C},$$

is called the characteristic function of the deterministic differential equation.

$$v_0 := \max\{Re\lambda \mid h(\lambda) = 0\} < \infty \text{ (top coefficient)},$$
$$v_{i+1} := \max\{Re\lambda \mid h(\lambda) = 0, \mathrm{Re}\,\lambda < v_i\}, \quad i \geq 0.$$

The fundamental solution $x_0(t)$:

$$\dot{x}_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$
$$x_0(t) = \mathbf{1}_{\{0\}}(t), \quad t \in [-r, 0].$$

The inverse Laplace transform yields for some $c < v_0$:

$$x_0(t) = p(t)\exp\{v_0 t\} + o(e^{ct}), \quad t \geq 0.$$

$p(t)$ polynomially bounded function of t.

$v_0 < 0$: x_0 tends exponentially to zero.

Back to the stochastic case

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)\,dt + dB(t), \quad t \geq 0 \tag{2}$$

$$X(u) = \xi(u), \quad t \in [-u, 0]. \tag{2'}$$

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- There exists a stationary solution if and only if $v_0 < 0$.
- This stationary solution is centered, Gaussian with covariance function

$$K(h) = \int_0^\infty x_0(s+h)x_0(s)\,ds, \quad h \geq 0.$$

- It satisfies the analogue of the Yule-Walker equation

$$\dot{K}(t) = \int_{-r}^{0} K(t+u)a(du),\ t > 0.$$

Back to the stochastic case

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- There exists a stationary solution if and only if $v_0 < 0$.
- This stationary solution is centered, Gaussian with covariance function

$$K(-h) = K(h) = \int_0^\infty x_0(s+h)x_0(s)\,ds, \quad h \geq 0.$$

- If $v_0 < 0$, then the covariance function decreases exponentially to zero as $\exp[v_0 h]$ for $h \to \infty$.

"Short-range dependence"

Statistical questions:

$$dX(t) = \int_{-r}^{0} X(t+s)a(ds)\,dt + dB(t), \quad t \geq 0,$$
$$X(u) = \xi(u), \quad u \in [-r, 0].$$

Likelihood process:

$$L_t = \frac{d\,\mathbb{P}_a^t}{d\,\mathbb{P}_0^t},$$

Log-likelihood process:

$$\ell_t = \log L_t = \int_{-r}^{0} \underbrace{\Big(\int_{0}^{t} X(s+u)dX(s)\Big)}_{V_t} a(du) - \frac{1}{2}\int_{-r}^{0}\Big(\int_{-r}^{0} \underbrace{\int_{0}^{t} X(s+u)X(s+v)ds}_{I_t}\, a(du)\Big)\, a(dv),$$
$$= \langle V_t, a\rangle - \frac{1}{2}\langle I_t a, a\rangle \to \max.$$

Obviously

- $V_t(\cdot) \in C([-r,0])$ and $I_t(\cdot,\cdot) \in C([-r,0]^2)$,
- I_t defines a Hilbert-Schmidt operator from $BV([-r,0])$ to $C([-r,0])$ and we call it the Fisher information operator.

The maximum likelihood equation

$$I_t \hat{a}_t(\cdot) = V_t(\cdot), \quad \text{i.e.,}$$

$$\int_{-r}^{0} \left(\int_{0}^{t} X(s+\cdot) X(s+u)\, ds \right) \hat{a}_t(du) = \int_{0}^{t} X(s+\cdot)\, dX(s).$$

$\hat{a}_t$ ... nonparametric maximum likelihood estimator.
I_t is a compact linear operator, thus the ML problem is an ill-posed problem.

Nonparametric case: Nonparametric estimation for $a(\cdot)$ based on the continuous observation of $(X(t), t \in [0, T])$ see Reiss (2002), [15].

Parametric case:

$$dX(t) = \int_{-r}^{0} X(t+s) a_\theta(ds)\, dt + dB(t), \quad t \geq 0, \tag{3}$$

$$X(s) = \xi(s), \quad s \in [-r, 0]. \tag{3'}$$

Aim

- Estimate θ based on continuous observation $X(t)$, $t \in [0, T]$.
- Study asymptotic properties of the estimator.

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- Estimate θ based on continuous observation $X(t)$, $t \in [0, T]$.
- Study asymptotic properties of the estimator.

Example (Gushchin, K. (1999), [5])

$$dX(t) = [aX(t) + bX(t-r)]\,dt + dB(t),$$
$$\theta = (a,b) \in \mathbb{R}^2.$$

Maximum likelihood estimator $\hat{\theta}(T)$:

$$I_T^{-1} \cdot V_T = \begin{pmatrix} \int_0^T X^2(t)dt & \int_0^T X(t)X(t-r)dt \\ \bullet & \int_0^T X^2(t-r)\,dt \end{pmatrix}^{-1} \begin{pmatrix} \int_0^T X(t)\,dX(t) \\ \int_0^T X(t-r)\,dX(t) \end{pmatrix}$$

and

$$\hat{\theta}_T - \theta = I_T^{-1} V_T^{0)} \qquad \text{with}$$

$$V_T^0 = \begin{pmatrix} \int_0^T X(s)\,dB(s) \\ \int_0^T X(s-r)\,dB(s) \end{pmatrix}.$$

Example (Gushchin, K. (1999), [5])

$$dX(t) = [aX(t) + bX(t-r)]\,dt + dB(t),$$
$$\theta = (a,b) \in \mathbb{R}^2.$$

There are eleven different cases of asymptotic behaviour of $\hat{\theta}_T$, depending on $\theta = (a,b)^T$:

- in one case N: LAN-property,
- in three cases M_1, M_2, M_3: LANM-property,
- in five cases $Q_1, Q_2, \ldots, Q_5$: LAQ-property,
- in two cases P_1, P_2: PLAMN-property.

Both, the corresponding sets of parameters and the limit distributions are calculated explicitly.
In this basic example: LAN-property holds iff $v_0 < 0$, i.e., iff there exists a stationary solution.

The LAN-property

It arises the question, under which general assumptions on $\theta \mapsto a_\theta$ the LAN-property holds.

$$dX(t) = \int_{-r}^{0} X(t+s)a_\theta(ds)\,dt + dB(t), \tag{3}$$

$$X(u) = \xi(u), \quad u \in [-r,0]. \tag{3'}$$

- $\theta \in \Theta \subseteq \mathbb{R}^k$, Θ open,
- $M_s = \{a :\ \text{(3) admits a stationary solution}\}$,
- Assume $\mathscr{A} := \{a_\theta(ds) : \theta \in \Theta \subseteq \mathbb{R}^k\} \subseteq M_s$.

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Localization around a fixed $\theta \in \Theta$:

$$\eta = \theta + \varphi_T(\theta)u, \qquad u \in \mathbb{R}^k$$

where $\varphi_T(\theta)$ is a $k \times k-$ regular matrix, with

$$\varphi_T(\theta) \to 0 \text{ for } T \to \infty \qquad \text{(normalizing matrix)}$$

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Assumption (A): (Condition on regularity)

1) $\exists\, C < \infty : \|a_\theta - a_\eta\|_v \leq C\|\theta - \eta\|$,
2) $\theta \mapsto \int_{[-r,0]} g(u)a_\theta(du)$, continuously differentiable in Θ with the gradient $\int_{[-r,0]} g(u)\dot{a}_\theta(du)$ for all $g \in C[-r,0]$, $\dot{a}_\theta = (\dot{a}_\theta^{(1)}, \dot{a}_\theta^{(2)}, \dots, \dot{a}_\theta^{(k)})^T$, $\theta \in \Theta$,
3) $\mathscr{A} = (a_\theta,\, \theta \in \Theta)$, Θ open, bounded subset of $\mathbb{R}^k$
4) $a_\theta \in M_s,\ \theta \in \bar{\Theta}$,
5) $a_\theta \neq a_\eta, \quad \theta \in \Theta,\ \eta \in \bar{\Theta},\ \theta \neq \eta$.

Example

$$dX(t) = \theta X(t-r)dt + dB(t).$$

Assumption (A) holds

$$\int_{-r}^{0} g(u) a_\theta(du) = \theta g(-r),$$

$$a_\theta(\cdot) = \theta \delta_{-r}, \quad \dot{a}_\theta(\cdot) = \delta_{-r}.$$

Example

$$dX(t) = bX(t-\theta)\, dt + dB(t)$$

does not satisfy assumption (A) because

$$\theta \mapsto \int_{[-r,0]} g(u) a_\theta(\, du) = bg(-\theta)$$

is not differentiable for all $g \in C[-r,0]$.

Theorem (Gushchin, K. (2003), [6])

Under the assumption (A) for every compact set $K \subseteq \Theta$ it holds uniformly in $\theta \in K$:

- $$\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{D} N(0, \Sigma^{-1}(\theta)), \quad T \longrightarrow \infty,$$

  where

  $$\Sigma_{ij}(\theta) = \int\int_{\mathscr{J} \times \mathscr{J}} K_\theta(u - v)\dot{a}_{\theta,i}(du)\dot{a}_{\theta,j}(dv).$$

- *All moments of $\sqrt{T}(\hat{\theta}_T - \theta)$ under $\mathbb{P}_T^\theta$ tend as $T \longrightarrow \infty$ to the corresponding moments of a normal distribution with parameters $(0, \Sigma^{-1}(\theta))$.*
- *The maximum likelihood estimator $\hat{\theta}_T$ is asymptotically efficient in K.*

As an intermediate step it was proved

$$
\begin{aligned}
Z_{T,\theta}(u) &= \frac{d\,\mathbb{P}^T_{\theta+\varphi_T u}}{d\,\mathbb{P}^T_\theta} \\
&= \exp\Big\{ \int_0^T \int_{-r}^0 X(s+v)\left(a_{\theta+\varphi_T u}(dv) - a_\theta(dv)\right) dX(s) \\
&- \frac{1}{2}\int_0^T \Big(\int_{-r}^0 \int_{-r}^0 X(s+v)X(s+w)\Big[a_{\theta+\varphi_T u}(dv)a_{\theta+\varphi_T u}(dw) \\
&- a_\theta(dv)a_\theta(dw)\Big]\Big) ds \Big\} \\
&\xrightarrow{T\to\infty} \exp\left\{ \langle Z, u\rangle - \frac{\|u\|^2}{2} \right\}, \qquad \text{(Local asymptotic normality)}
\end{aligned}
$$

where $Z \sim N(0, J_k)$.

Example

$$dX(t) = bX(t-\theta)dt + dB(t), \quad b < 0, \text{ fixed},$$

$$\theta \in (0, \frac{1}{e|b|}) =: \Theta \quad (\text{ensures } v_0 < 0).$$

$$Z_{T,\theta}(u) := \frac{d\,\mathbb{P}_{\theta+\varphi_T u}}{d\,\mathbb{P}_\theta}$$
$$= \exp\left(b \int_0^T (X(t-\theta-\varphi_T u) - X(t-\theta))\, dX(t) \right.$$
$$\left. - \frac{b^2}{2} \int_0^T (X(t-\theta-\varphi_T u) - X(-\theta))^2\, dt \right).$$

Lemma (K., Kutoyants (2000), [11])

For $\varphi_T = b^2 T^{-1}$ *the marginal distributions of* $Z_{T,\theta+\varphi_T u}(.)$ *converge to the marginal distributions of*

$$Z(u) = \exp\left\{\widetilde{B}(u) - \frac{1}{2}|u|\}\right\}, \quad u \in \mathbb{R}^1,$$

uniformly over every compact set $K \subseteq \mathbb{R}^1$.

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Remember:

$a_\theta(dv) = b\,\delta_{-\theta}(dv)$:

$$Z_T(u) = \frac{d\,\mathbb{P}^T_{\theta+\varphi_T(\theta)u}}{d\,\mathbb{P}^T_\theta} \longrightarrow Z(u) = \exp[\tilde{B}(u) - \frac{1}{2}|u|].$$

$a_\theta(dv) = \theta\,\delta_{-1}(dv)$:

$$Z_T(u) = \frac{d\,\mathbb{P}^T_{\theta+\varphi_T(\theta)u}}{d\,\mathbb{P}^T_\theta} \longrightarrow Z(u) = \exp[uB - \frac{1}{2}u^2].$$

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Lemma (Gushchin, K. (2008), [7])

For every $H \in [\frac{1}{2}, 1]$ one can find examples $a_\theta(dv)$ with

$$\frac{d\,\mathbb{P}^T_{\theta+\varphi_T(\theta)u}}{d\,\mathbb{P}^T_\theta} \longrightarrow Z(u) = \exp\left[B^H(u) - \frac{\mathbb{E}(B^H(u))^2}{2}\right],$$

where B^H is a fractional Brownian motion with Hurst parameter H.

$H = \frac{1}{2}$: $B^H(u)$ *Brownian motion*

$H = 1$: $B^H(u) = uB,\ B \sim N(0,1)$.

Fractional calculus: an introduction

Riemann-Liouville Integral

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt.$$

Riemann-Liouville Derivative

$$D^{\alpha} f := D^1 J^{1-\alpha} f, \quad \alpha \in (0,1).$$

Caputo Derivative

$$D_c^{\alpha} f := D^{\alpha}[f - f(0)].$$

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Fractional affine DDE

$$D_0^\alpha y(t) = \int_{-r}^{0} y(t+s)a(ds) + f(t), \tag{5}$$

$$y(s) = \xi(s), \quad s \in [-r, 0], \quad \xi(\cdot) \in C([-r, 0]). \tag{5'}$$

Solution of (5, 5'):

$$y(t) = \xi(0)x_0(t) + D^{1-\alpha} \int_{-r}^{0} \left(\int_{u}^{0} x_0(t+u-v)\xi(v)dv \right) a(du) + \int_{0}^{t} x_0(t-s)f(s)ds.$$

Here $x_0(\cdot)$ denotes the fundamental solution of (5, 5'), i.e.,

$$\begin{aligned} D_0^\alpha x_0(t) &= \int_{-r}^{0} x_0(t+s)a(ds), \\ x_0(s) &= \mathbf{1}_{\{0\}}(s), \quad s \in [-r, 0]. \end{aligned}$$

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Fundamental solution

$$D_0^\alpha x_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$
$$x_0(s) = \mathbf{1}_{\{0\}}(s), \quad s \in [-r,0].$$

Laplace-transform:

$$\hat{x}_0(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^\alpha - \int_{-r}^{0} e^{\lambda s} a(ds)}, \; Re\,\lambda > \|a\|^{\frac{1}{\alpha}}.$$

Characteristic function:

$$h_\alpha(\lambda) = \left[\lambda^\alpha - \int_{-r}^{0} e^{\lambda s} a(ds)\right] \cdot \lambda^{1-\alpha} = \lambda - \lambda^{1-\alpha} \int_{-r}^{0} e^{\lambda s} a(ds).$$

Fundamental solution

$$D_0^\alpha x_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$

$$x_0(s) = \mathbf{1}_{\{0\}}(s), \quad s \in [-r,0].$$

There is a qualitative jump from $\alpha = 1$ to $\alpha < 1$:
The characteristic function

$$h_\alpha(\lambda) = \lambda - \lambda^{1-\alpha} \int_{-r}^{0} \mathrm{e}^{\lambda s} a(ds)$$

is

- holomorphic only on $\mathbb{C} \setminus \mathbb{R}_-$,
- discontinuous on $\mathbb{R}_-$,

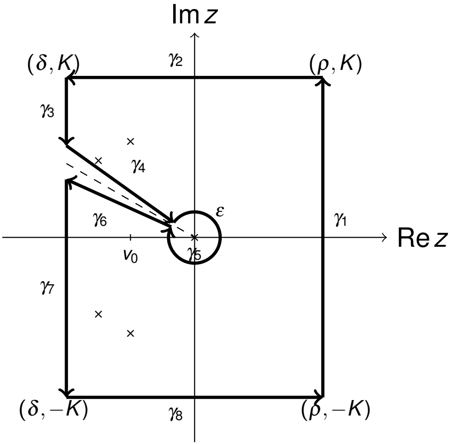

Figure: Contour integration of $\chi_\lambda^{-1}(z)e^{zt}$.

Fundamental solution

$$D_0^\alpha x_0(t) = \int_{-r}^{0} x_0(t+s)a(ds),$$
$$x_0(s) = \mathbf{1}_{\{0\}}(s), \quad s \in [-r,0].$$

Theorem (Krol (2008), [10])

For $v_0 < 0$ we have $\quad x_0(t) = p(t)e^{v_0 t} + g(t),\ t \geq 0,$

with $p(t)$ polynomially bounded function and

$$g(t) = \begin{cases} O(t^{-\alpha}), & \text{if } a([-r,0]) \neq 0, \\ 1 + O(t^{-k_0+\alpha}), & \text{if } a([-r,0]) = 0, \text{ for some } k_0 \in \mathbb{N}. \end{cases}$$

Thus the asymptotic of the fundamental solution $x_0(t)$ for $t \to \infty$ is not exponentially but polynomially.

Stochastic case

$$X(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_{-r}^0 X(s+u)a(du)ds + B(t),$$
$$X(s) = \xi(s), \quad s \in [-r,0].$$

Theorem (Krol (2008), [10])

There exists a stationary solution iff $v_0 < 0$, $a([-r,0]) \neq 0$ *and* $\alpha > \frac{1}{2}$.

Stochastic case

$$
\begin{aligned}
X(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_{-r}^0 X(s+u)a(du)ds + B(t), \\
X(s) &= \xi(s), \quad s \in [-r, 0].
\end{aligned}
$$

Corollary

The covariance function $K(t)$ of the stationary solution is given by

$$
K(t) = \int_0^t x_0(s)\, x_0(s+t)ds.
$$

$K(t)$ tends polynomially to zero for $t \to \infty$.

Thus, $(X(t), t \geq 0)$ has

"long-range dependence"

Thank you for your attention!

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