Example of LDP approach to smoothing problem

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Outline

1. The Constant Elasticity of Variance Model (CEV)

CEV model, $\boxed{\gamma < 1,\ B_t\text{-Brownian motion}}$,

$$dX_t = \mu X_t dt + \varepsilon X_t^{\gamma} dB_t, \quad X_0^{\varepsilon} = x > 0,$$

- was introduced by Cox (1996)
- was applied in Option Pricing Models (Delbae & Shirakawa (www.math.ethz.ch/ delbaen), Lu & Hsu (2005).

In view of $\gamma < 1$, CEV model is characterized by the following property: for

- any $T > 0$
- the time to "ruin" $\tau_0 = \inf\{t : X_t = 0\}$

$$\mathrm{P}(\tau_0 \le T) > 0.$$

2. $\gamma \in \left[\frac{1}{2},\ 1\right)$.

For $\gamma = \frac{1}{2}$, the generator

$$\mathfrak{L} = x\left[\mu\frac{\partial}{\partial x} + \frac{\varepsilon^2}{2}\frac{\partial^2}{\partial x^2}\right]$$

of Markov process X_t indicates that X_t is “Branching Diffusion”.
We avoid the case of $\gamma < \frac{1}{2}$ in order to make applicable Yamada-Watanabe theorem providing the unique solution of stochastic differential equation.

3. Main problem

For fixed $x = X_0$ and $T > 0$, to create a reasonable estimate $(\widehat{u}_t)_{t\in[0,T]}$, $\widehat{u}_0 = x$, of $(X_t)_{t\in[0,\tau_0\wedge T]}$ in the following setting: for sufficiently small ε, δ

$$\mathsf{P}\Big(\sup_{t\in[0,T]}|X_t - \widehat{u}_t| \le \delta\Big) \gtrsim \mathsf{P}\Big(\sup_{t\in[0,T]}|X_t - u_t| \le \delta\Big)$$

for any continuous function $(u_t)_{t\in[0,T]}$, $u_0 = x$ absorbed at the point $\theta(u) = \inf\{t : u_t = 0\} \le T$.
More precisely,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\in[0,T]}|X_t - \widehat{u}_t| \le \delta\Big) \ge \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\varepsilon^2\log\mathsf{P}\Big(\sup_{t\in[0,T]}|X_t - u_t| \le \delta\Big).$$

4. Main tools

1) With $\alpha < x$ and $\tau_\alpha = \inf\{t : X_t = \alpha\}$, $\boxed{\gamma < 1}$ enables us to apply the Itô formula to $X^{1-\gamma}_{\tau_\alpha\wedge t} e^{-(1-\gamma)\mu(\tau_\alpha\wedge t)}$ and obtain:

$$
\begin{aligned}
&X^{1-\gamma}_{\tau_\alpha\wedge T} e^{-(1-\gamma)\mu(\tau_\alpha\wedge T)} \\
&= x^{1-\gamma} - \int_0^{\tau_\alpha\wedge T} \frac{\varepsilon^2}{2}\gamma(1-\gamma)\frac{e^{(1-\gamma)\mu s}}{X_s^{1-\gamma}} ds \\
&+ \int_0^{\tau_\alpha\wedge T} \varepsilon(1-\gamma)e^{-(1-\gamma)\mu s} dB_s.
\end{aligned}
$$

2) $M_t = \int_0^t (1-\gamma)e^{-(1-\gamma)\mu s} dB_s$ Gaussian continuous martingale with the predictable quadratic variation process

$$
\langle M\rangle_t = \int_0^t (1-\gamma)^2 e^{-2(1-\gamma)\mu^2 s} ds.
$$

5. Main tools

3) $\lim_{\alpha\to 0}\tau_\alpha = \tau_0$

4) By martingale convergence theorem

$$\lim_{\alpha\to 0} M_{\tau_\alpha\wedge T} = M_{\tau_0\wedge T}$$

5) Set $V_\alpha = \int_0^{\tau_\alpha\wedge T}\frac{\varepsilon^2}{2}\gamma(1-\gamma)\frac{e^{(1-\gamma)\mu s}}{X_s^{1-\gamma}}ds$. By monotone convergence theorem

$$\lim_{\alpha\to 0} V_\alpha = V_0 = \int_{[0,\tau_0\wedge T)}\frac{\varepsilon^2}{2}\gamma(1-\gamma)\frac{e^{(1-\gamma)\mu s}}{X_s^{1-\gamma}}ds.$$

6) $X_{\tau_0\wedge T}^{1-\gamma}e^{-(1-\gamma)\mu(\tau_0\wedge T)} = x^{1-\gamma} - V_0 + \varepsilon M_{\tau_0\wedge T}$.

6. Lower bound of $\mathrm{P}(\tau_0 \le T)$

6) $V_0 > 0$ provide

$$\{\tau_0 > T\} = \{x^{1-\gamma} + \varepsilon M_T > 0\} \cap \{\tau_0 > T\},$$

that is,

$$\{\tau_0 \le T\} = \{x^{1-\gamma} + \varepsilon M_T \le 0\} \cup \{\tau_0 \le T\} \supseteq \{-\varepsilon M_T \ge x^{1-\gamma}\}.$$

Hence, in view of $-\varepsilon M_T$ is zero mean Gaussian r.v. with the variance $\langle M\rangle_t$, (here $\varPhi$ is (0,1)-Gaussian distribution)

$$\mathrm{P}(\tau_0 \le T) \ge 1 - \varPhi\Big(\sqrt{\varepsilon^2 \langle M\rangle_T} x^{1-\gamma}\Big).$$

7. Lower bound of $\mathrm{P}(\tau_0 \le T)$ in the logarithmic scale

The gaussianity of M_T with $\mathrm{E}M_T = 0$ provide

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathrm{P}(-\varepsilon M_T \ge x^{1-\gamma}) = -\frac{x^{2(1-\gamma)}}{2\langle M \rangle_T},$$

while the inclusion implies $\{\tau_0 \le T\} \supseteq \{-\varepsilon M_T \ge x^{1-\gamma}\}$

$$\varliminf_{\varepsilon \to 0} \varepsilon^2 \log \mathrm{P}(\tau_0 \le T) \ge -\frac{x^{2(1-\gamma)}}{2\langle M \rangle_T}.$$

8. Upper bound of $\mathrm{P}(\tau_0 \le T)$ in the logarithmic scale

Unfortunately, a computation of upper bound for $\mathrm{P}(\tau_0 \le T)$, coinciding with the lower one, might be computed by applying Large Deviation technique for family $\{(X_t)_{t\in[0,T]}\}_{\varepsilon\to 0}$:

- speed rate ε^2
- rate function

$$J_T(u) = \begin{cases} \frac{1}{2\sigma^2}\int_0^T \Big(\frac{\dot{u}_t - \mu u_t}{u_t^\gamma}\Big)^2 I_{\{u_t>0\}}dt, & {du_t=\dot{u}_t dt \atop u_0=x} \\ \infty, & \text{otherwise.} \end{cases}$$

9. How to apply LD

Introduce a closed (in the uniform metric) set

$$\mathrm{D} = \Big\{u = (u_t)_{t\in[0,T]} : u_0 = x; \theta(u) = \inf\{t : u_t = 0\} \le T\Big\}$$

and use

$$\{\tau_0 \le T\} = \{(X_t)_{t\in[0,T]} \in \mathrm{D}\}$$

since by main LD property

$$\varlimsup_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(\tau_0 \le T) \le -\min_{u\in \mathrm{D}} J_T(u)$$

The function $u \in \mathrm{D}$ is absorbed at $\theta(u) \le T$. Hence $\min_{u\in\mathrm{D}} J_T(u)$ is replaced by $\min_{u\in\mathrm{D}} J_{\theta(u)\wedge T(u)}$

10. Minimization problem

For $t < \theta(u)$, set

$$w_t = \frac{\dot{u}_t - \mu u_t}{u_t^\gamma},$$

that is,

$$\dot{u}_t = \mu u_t + \sigma u_t^\gamma w_t, \ u_0 = x.$$

From all functions w_t's, we have to chose one $\widehat{w}_t$ (hereafter $\widehat{u}_t$ corresponds to $\widehat{w}_t$) such that :

$$\int_0^{\theta(\widehat{u})\wedge T} \widehat{w}_t^2 dt \le \int_0^{\theta(u)\wedge T} w_t^2 dt.$$

11. Minimization problem

The function $\widehat{w}_t$ can be readily found (!) since the function $v_t = u_t^{1-\gamma}$ solves a linear differential equation

$$\dot{v}_t = \mu(1-\gamma)v_t + \sigma(1-\gamma)w_t, \; v_0 = x^{1-\gamma}$$

and

$$v_{\theta(u)} = 0 \text{ in view of } u_{\theta(u)} = 0.$$

Write

$$0 = v_{\theta(u)} = e^{\mu(1-\gamma)\theta(u)}\Big[x^{1-\gamma} + \sigma(1-\gamma)\int_0^{\theta(u)} e^{-\mu(1-\gamma)s} w_s ds\Big],$$

or, equivalently, $-\frac{x^{1-\gamma}}{\sigma(1-\gamma)} = \int_0^{\theta(u)} e^{-\mu(1-\gamma)s} w_s ds.$

12. Minimization problem

The Cauchy-Schwarz's inequality implies

$$\int_0^{\theta(u)} w_t^2 dt \geq \frac{2\mu x^{2(1-\gamma)}}{\sigma^2(1-\gamma)[1-e^{-2\mu(1-\gamma)\theta(u)}]}$$
$$\geq \frac{2\mu x^{2(1-\gamma)}}{\sigma^2(1-\gamma)[1-e^{-2\mu(1-\gamma)T}]} = \frac{1}{\langle M\rangle_T}.$$

This gives a hint to choose
- $\theta(\widehat{u}) = T$
- $\widehat{w}_t = \text{const}.e^{-\mu(1-\gamma)t}$
- "const" such that

$$\int_0^T \widehat{w}_t^2 dt = \frac{1}{\sigma^2}\frac{2\mu x^{2(1-\gamma)}}{(1-\gamma)[1-e^{-2(1-\gamma)\mu T}]} = \frac{x^{2(1-\gamma)}}{\langle M\rangle_T}$$

13. Minimization problem

In particular,

$$\frac{1}{2}\int_0^T \widehat{w}_t^2 dt = \frac{x^{2(1-\gamma)}}{2\langle M\rangle_T},$$

that is,

$$\varlimsup_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(\tau_0 \le T) \le -\frac{x^{2(1-\gamma)}}{2\langle M\rangle_T}.$$

Thus,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathrm{P}(\tau_0 \le T) = -\frac{x^{2(1-\gamma)}}{2\langle M\rangle_T}.$$

14. $\widehat{u}_t$ - optimal smoother

The validity of

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\varepsilon^2\log \mathrm{P}\Big(\sup_{t\in[0,T]}|X_t-\widehat{u}_t|\le\delta\Big) \ge \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\varepsilon^2\log \mathrm{P}\Big(\sup_{t\in[0,T]}|X_t-u_t|\le\delta\Big).$$

follows from

$$-J_T(\widehat{u})=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\varepsilon^2\log \mathrm{P}\Big(\sup_{t\in[0,T]}|X_t-\widehat{u}_t|\le\delta\Big)$$

$$-J_T(u)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\varepsilon^2\log \mathrm{P}\Big(\sup_{t\in[0,T]}|X_t-u_t|\le\delta\Big)$$

and

$$J_T(\widehat{u})\le J_T(u).$$

15. Explicit formula for $\widehat{u}_t$

$$\begin{cases} \widehat{v}_0 = x^{1/\gamma} \\ \dot{\widehat{v}}_t = \mu(1-\gamma)\widehat{v}_t + \sigma(1-\gamma)\widehat{w}_t, \\ \widehat{w}_t = -\frac{1}{\sigma}\frac{2\mu}{1-e^{-2\mu(1-\gamma)T}}e^{-\mu(1-\gamma)t} \end{cases}$$

imply

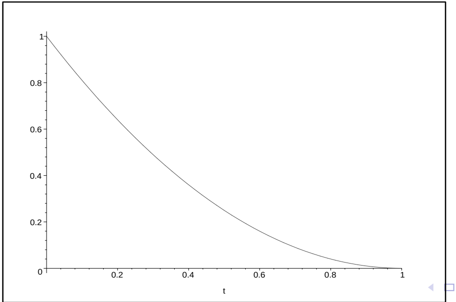
$$\widehat{v}_t = e^{\mu(1-\gamma)t}\left[x^{1/1-\gamma} - \frac{1-e^{-2\mu(1-\gamma)t}}{1-e^{-2\mu(1-\gamma)T}}\right]$$

and, in turn,

$$\widehat{u}_t = e^{\mu t}\left[x^{1/1-\gamma} - \frac{1-e^{-2\mu(1-\gamma)t}}{1-e^{-2\mu(1-\gamma)T}}\right]^{1/1-\gamma}.$$

16. Example 1

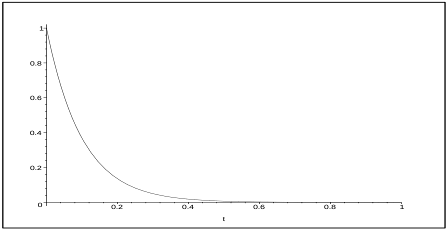
$$\begin{cases} x = 1 \\ \mu = 0 \\ \gamma = \frac{1}{2} \\ T = 1 \end{cases} \Longrightarrow \widehat{u}_t = \lim_{\mu \to 0} \sqrt{\frac{1 - e^{-2\mu(1-\gamma)t}}{1 - e^{-2\mu(1-\gamma)}}}. = \sqrt{1 - t}$$



17. Example 2

$$\begin{cases} x = 1 \\ \mu = \pm 10 \\ \gamma = \frac{1}{2} \\ T = 1 \end{cases} \quad \Longrightarrow \widehat{u}_t = e^{\pm 10t}\left[1 - \frac{1 - e^{\mp 10t}}{1 - e^{\mp 10}}\right]^{1/2}.$$

For $\mu = 10$ and $\mu = -10$, functions $\widehat{u}_t$'s coincide.



18. Large Deviation Principle discussion

Formally, CEV model

$$dX_t = \mu X_t dt + \varepsilon X_t^\gamma dB_t, \quad X_0^\varepsilon = x > 0,$$

is in a framework of Freidlin-Wentzell's setting if one may forget on singular and non-Lipschitz "diffusion" parameter x^γ.

However, the Freidlin LDP result remains valid with slightly modified rate function

$$J_T(u) = \frac{1}{2}\int_0^T \frac{[\dot{u}_t - \mu u_t]^2}{u^{2\gamma}} I_{\{u_t>0\}} dt$$

instead of

$$J_T(u) = \frac{1}{2}\int_0^T \frac{[\dot{u}_t - \mu u_t]^2}{u^{2\gamma}} dt$$

19. Large Deviation Principle discussion

For $\gamma = \frac{1}{2}$, the LDP was published by

Donati-Martin, C.; Rouault, A.; Yor, M.; Zani, M. Large deviations for squares of Bessel and Ornstein-Uhlenbeck processes. *Probab. Theory Related Fields.* 129 (2004), no. 2, 261–289.

We show that the LDP holds true for $\gamma \in \left[\frac{1}{2}, 1\right)$ and can be proved by applying Puhalskii's approach.

(i) local LDP: for any nonnegative continuous function u_t absorbing at zero,

$$\overline{\lim_{\delta \to 0}}\ \overline{\lim_{\varepsilon \to 0}}\ \varepsilon^2 \log \mathrm{P}\left(\sup_{t \in [0,T]} \left| X_t^\varepsilon - u_t \right| \le \delta \right) \le -J(u)$$

$$\underline{\lim_{\delta \to 0}}\ \underline{\lim_{\varepsilon \to 0}}\ \varepsilon^2 \log \mathrm{P}\left(\sup_{t \in [0,T]} \left| X_t^\varepsilon - u_t \right| \le \delta \right) \ge -J(u)$$

20.

(ii) Exponential tightness:

$$\lim_{C\to\infty}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon^2\log \mathrm{P}\Big(\sup_{t\le T} x^\varepsilon_t \ge C\Big) = -\infty,$$

$$\lim_{\Delta\to 0}\overline{\lim_{\varepsilon\to 0}}\sup_{\vartheta\le T}\varepsilon^2\log \mathrm{P}\Big(\sup_{t\le \Delta}|x^\varepsilon_{\vartheta+t} - x^\varepsilon_\vartheta| \ge \eta\Big) = -\infty,\ \forall\,\eta>0,$$

where ϑ is a stopping time relative to the filtration generated by Brownian motion $(B_t)_{t\ge 0}$.

Conditions (i) and (ii) are verified with the help of two facts:
1) [Liptser and Spokoiny] Let N_t be a continuous local martingale, $N_0 = 0$, with predictable variation process $\langle N \rangle_t$. Then for any $\eta > 0$, $l > 0$, and any measurable set $\mathfrak{A}$,

$$\mathrm{P}\Big(\sup_{t\in[0,T]} |N_t| \ge \eta, \langle N \rangle_T \le l,\ \mathfrak{A}\Big) \le 2e^{-\frac{\eta^2}{2l}}.$$

2) [Dupuis and Ellis] Let $u = (u_t)_{t\in[0,T]}$ be an absolutely continuous function mapping of $[0, T]$ into $\mathbb{R}$. Then for each real number a the set $\{t : u_t = a, \dot{u}_t \neq 0\}$ has Lebesgue measure zero
and some "regularization techniques".

Theorem. *The family* $\{(X_t^\varepsilon)_{t\in[0,T]}\}_{\varepsilon\to 0}$ *obeys the LDP in the metric space* $(\mathbb{D}_{[0,T]}(\mathbb{R}_+), \varrho)$ *(ϱ is the uniform metric) with the rate speed* $\boxed{\varepsilon^2}$ *and the rate function* $(u = (u_t)_{t\in[0,T]} \in \mathbb{A}_{[0,T]})$

$$J(u) = \begin{cases} \frac{1}{2}\int_0^{T\wedge\theta(u)} \frac{(\dot{u}_t - \mu u_t)^2}{u_t^{2\gamma}} I_{\{u_t>0\}}\,dt, & \substack{u_0=x \\ du_t=\dot{u}_t dt} \\ \infty, & \text{otherwise} \end{cases}.$$

23. Conclusion.

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathrm{P}_x(\tau_0^\varepsilon \le T) = -J(\widehat{u})$$

$$= -\frac{1}{2}\begin{cases} \frac{2\mu x^{2(1-\gamma)}}{(1-\gamma)[1-e^{-2(1-\gamma)\mu T}]}, & \text{for } \mu \neq 0 \\ \frac{x^{2(1-\gamma)}}{(1-\gamma)^2 T}\Big), & \text{for } \mu = 0. \end{cases}$$

$\widehat{u}_t$ - the optimal smoother in a sense $J(\widehat{u}) \le J(u)$ for any u_t with $u_0 = x > 0$ and $\theta(u) = \inf\{t : u_t = 0\} \le T$.