EXPANSION OF THE ASYMPTOTICALLY CONDITIONALLY NORMAL LAW

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Menu

- Estimation of the unknown parameter in the diffusion coefficient
- Expansion of a martingale
- Expansion of the asymptotically conditionally normal law
- Non synchronous covariance estimator

Estimation of the diffusion coefficient

- Historically, the estimation of diffusion coefficients had been studied by theoretical statisticians in the context of estimation for stochastic differential equations.
- They proposed many estimators and investigated their asymptotic properties such as asymptotic normality and asymptotic mixed normality as well as consistency:

Estimation of the diffusion coefficient

Prakasa Rao (1983,1988),
Dacunha-Castelle and Florens-Zmirou (1986),
Florens-Zmirou (1989), Yoshida (1992, 2005),
Kessler (1997), Shimizu and Yoshida (2006),
Bibby and Soerensen (1995), Masuda (2005),
Genon-Catalot and Jacod (1993),
Soerensen (2000),
Soerensen and Uchida (2003),
Uchida (2003,2008)
among vast literature.

The role of the asymptotic expansion in statistics

Asymptotic expansion is the basis for major fields in statistics. Text books.

1. higher-order asymptotic inference: Akahira-Takeuchi (1981), Pfanzagl (1985), Siotani et al. (1985), Ghosh (1994), Barndorff-Nielsen and Cox (1994), Pace and Salvan (1995), Akahira and Takeuchi (1995), Kutoyants (1995, 2004)
2. resampling methods, bootstrap method: Hall (1992)
3. information geometry: Amari (1985), Amari et al. (1987), Murray and Rice (1993)
4. information criteria: Konishi and Kitagawa (2008)

Asymptotic expansion: normal limit

SDE

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sqrt{\theta}\sigma_s dw_s, \quad t \in [0,1].$$

- $\mathcal{B} = (\Box, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, P)$: **a stochastic basis,**
- $b_t, \ \sigma_t$: **Itô processes**
- $\{X_{t_i}, \sigma_{t_i}\}_{i=0,1,\dots,n}$ **data,** $t_i = \frac{i}{n}$.

  Ex. $\sigma_t = \sigma(X_t)$ **for some function** $\sigma(\cdot)$.

Asymptotic expansion: normal limit

A natural quasi-likelihood estimator of θ:

$$\hat{\theta}_n = \sum_{i=1}^{n} \left(\frac{X_{t_i} - X_{t_{i-1}}}{\sigma_{t_{i-1}}} \right)^2$$

Notation

$$\mathcal{X}_n := \frac{\sqrt{n}}{\sqrt{2}\theta} \left(\hat{\theta}_n - \theta \right).$$

$$\Delta w_i = w_{t_i} - w_{t_{i-1}},$$

Asymptotic expansion: normal limit

Stochastic expansion of $\mathcal{X}_n$:

$$\mathcal{X}_n = M_n + \frac{1}{\sqrt{n}} N_n,$$

where

$$M_n = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} H_2(\sqrt{n}\Delta w_i)$$

and

Asymptotic expansion: normal limit

$$N_n = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{3}\sigma_{t_{i-1}}^{[1]}}{\sigma_{t_{i-1}}} H_3(\sqrt{n}\Delta w_i) + \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \cdot \left(\frac{\sqrt{2}\sigma_{t_{i-1}}^{[1]}}{\sigma_{t_{i-1}}} + \frac{\sqrt{2} b_{t_{i-1}}}{\sqrt{\theta}\sigma_{t_{i-1}}} \right) H_1(\sqrt{n}\Delta w_i) + \sum_{i=1}^{n} \frac{1}{n} F_{t_{i-1}} + R_n$$

with $\|R_n\|_p = O(n^{-1/2})$ **for every** $p > 1$**, and**

Asymptotic expansion: normal limit

with F_t given by

$$F_t = \frac{\sqrt{2}\sigma_t^{[0]}}{2\sigma_t} + \frac{(\sigma_t^{[1]})^2}{2\sqrt{2}\sigma_t^2} + \frac{\sqrt{2}b_t^{[1]}}{2\sqrt{\theta}\sigma_t} + \frac{\sqrt{2}b_t^2}{2\theta\sigma_t^2}.$$

σ_t is assumed to have a decomposition

$$\sigma_t = \sigma_0 + \int_0^t \sigma_s^{[1]} dw_s + \int_0^t \sigma_s^{[0]} ds.$$

Expansion of a martingale

The principal part M_n is rather simple in this example but can be treated as a martingale.

The key steps to the asymptotic expansion:

(i) asymptotic expansion for martingales

(ii) a perturbation method

(iii) investigation of the first-order behaviors of the first and second order terms.

Expansion of a martingale

Consider a probablity space equipped with a differential calculus in Malliavin's sense, an integration-by-parts formula and the Sobolev spaces $\mathbb{D}_{p,\ell}$ normed by $\| \cdot \|_{p,\ell}$.

Expansion of a martingale

Random variable $X_n = M_n + r_n N_n$, where

- r_n: constants $\downarrow 0$,
- $M_n = M^n_{T_n}$: the terminal variable of a continuous martingale $M^n = (M^n_t)_{t\in[0,T_n]}$,
- $\sup_n \|M_n\|_{p,k+1} + \sup_n \|r_n^{-1}(\langle M^n\rangle_{T_n} - 1)\|_{p,k} + \sup_n \|N_n\|_{p,k+1} < \infty$ for any $p > 1$.
- $(M_n, r_n^{-1}(\langle M^n\rangle_{T_n} - 1), N_n) \to^d (Z, \xi, \eta)$

Expansion of a martingale

Asymptotic expansion:

$$p_n(z) = \phi(z) + \frac{1}{2} r_n \partial_z^2 \big(E[\xi|Z=z]\phi(z)\big) - r_n \partial_z \big(E[\eta|Z=z]\phi(z)\big).$$

$$\mathcal{E}(M,\gamma) := \{f : \ \textbf{measurable} \ ; |f(x)| \le M(1+|x|^{\gamma}) \ (\forall x)\}.$$

Expansion of a martingale

Theorem 1. (Y (1997)) Let $M, \gamma > 0$, $p > 1$ and $q' > 2/3$. Under the above conditions for $k = 4$ and mild conditions, one has

$$\left| E\big[f(X_n)\big] - \int f(z)p_n(z)dz \right| \lesssim r_n^{-q'} P[\sigma_{M_n} < s_n]^{1/p} + o(r_n)$$

as $n \to \infty$ uniformly in $f \in \mathcal{E}(M,\gamma)$. Here s_n are random variables such that $\sup_n E[s_n^{-k}] < \infty$ for any $k > 1$.

Remark 1. Y (2001) for martingale with jumps.

Asymptotic expansion: normal limit

Apply Theorem 1 to

$$\mathcal{X}_n = \frac{\sqrt{n}}{\sqrt{2}\theta}\left(\hat{\theta}_n - \theta\right) = M_n + \frac{1}{\sqrt{n}} N_n$$

to obtain the Edgeworth expansion

Asymptotic expansion: normal limit

Theorem 2. **(Y (1997))**

$$\sup_x \left| P\left[\frac{\sqrt{n}}{\sqrt{2}\theta}(\hat{\theta}_n - \theta) \le x\right] - P_n(x) \right| = o\left(\frac{1}{\sqrt{n}}\right),$$

where

$$P_n(x) = \Phi(x) + \frac{1}{\sqrt{n}}\left(\frac{1}{2}\alpha(1 - x^2) - \beta\right)\phi(x)$$

with $\alpha = 2\sqrt{2}/3$ **and** $\beta = E[H(1)]$ **for some functional** $H(1)$ **obtained by a martingale-problem method.**

Remark

Remark 2. The principal part M_n is so good but why is the Malliavin calculus necessary?

$- - - - -$ The small second order terms can destroy the regularity of the distribution of the principal part:

$$X_n = Z + o_p(r_n^m) \quad (\forall m > 0) \quad \textbf{with } Z \sim N(0,1)$$
$$\not\Rightarrow \sup_x |P[X_n \le x] - \Phi(x)| = o(r_n).$$

It is necessary to verify that the second order terms do not affect the principal part so much.

Expansion of the asymptotically conditionally normal law

Mixed normal limit.

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma(X_s, \theta) dw_s, \quad t \in [0,1].$$

- $\mathcal{B} = (\square, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, P)$: **a stochastic basis**
- $\{X_{t_i}\}_{i=0,1,\ldots,n}$ **data,** $t_i = \frac{i}{n}$.

Expansion of the asymptotically conditionally normal law

Theorem 3. (Genon-Catalot and Jacod (1993)) A naturally defined quasi-likelihood estimator $\hat{\theta}_n$ for θ is aymptotically mixed normal:

$$\sqrt{n}(\hat{\theta}_n - \theta) \to^d G^{\frac{1}{2}}\zeta,$$

where G is $\mathcal{F}_1$-measurable $\perp\!\!\!\perp$ $\zeta \sim N(0,1)$.

Expansion of the asymptotically conditionally normal law

- Normal limit (CLT)

  $\Longrightarrow$ Asymptotic expansion

- Mixed normal limit

  $\Longrightarrow$?

Expansion of the asymptotically conditionally normal law

Observation.

- **Stable convergence**

$$Z_n \to^{d_s} G^{\frac{1}{2}}\zeta, \quad \zeta \sim N(0,1) \perp\!\!\!\perp \mathcal{F}_1$$

and $G_n \to^p G$.

- **Joint convergence**

$$(Z_n, G_n) \to^d (G^{\frac{1}{2}}\zeta, G)$$

Expansion of the asymptotically conditionally normal law

- **Lifting**

$$\mathcal{L}\{Z_n | G_n = x\} \Rightarrow N(0, x)$$

Remark 3. Y (2003) for the lifting.

Expansion of the asymptotically conditionally normal law

$(- \ ,\mathcal{F},\mathbf{F}=(\mathcal{F}_t)_{t\in[0,1]},P)$ a stochastic basis

$\mathcal{F}=\mathcal{F}_1$

d-dimensional $\quad Z_n = M_n + W_n + r_n N_n$

$M_n = M^n_{T_n}$ martingale

d_1-dimensional $\quad F_n$

Expansion of the asymptotically conditionally normal law

$(- \ ,\mathcal{F},\mathbf{F}=(\mathcal{F}_t)_{t\in[0,1]},P)$ a stochastic basis

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Expansion of the asymptotically conditionally normal law

$$
\begin{aligned}
M_\infty &= M_1^\infty \\
C_t^n &= \langle M^n \rangle_t \\
C_n &= \langle M^n \rangle_1 \\
\overset{\circ}{C}_n &= r_n^{-1}(C_n - C_\infty) \\
\overset{\circ}{W}_n &= r_n^{-1}(W_n - W_\infty) \\
\overset{\circ}{F}_n &= r_n^{-1}(F_n - F_\infty)
\end{aligned}
$$

Expansion of the asymptotically conditionally normal law

We assume

(i) $(C^n_\cdot, W_n, F_n) \to^p (C^\infty_\cdot, W_\infty, F_\infty)$.

(ii) $(M^n_\cdot, N_n, \mathring{C}_n, \mathring{W}_n, \mathring{F}_n) \to^{d-s(\mathcal{F})} (M^\infty_\cdot, N_\infty, \mathring{C}_\infty, \mathring{W}_\infty, \mathring{F}_\infty)$.

(iii) $\mathcal{L}\{M^\infty_t | \mathcal{F}\} = N_d(0, C^\infty_t)$.

In particular,

$$(M^n_\cdot, N_n, \mathring{C}_n, \mathring{W}_n, \mathring{F}_n, C^n_\cdot, W_n, F_n) \to^d (M^\infty_\cdot, N_\infty, \mathring{C}_\infty, \mathring{W}_\infty, \mathring{F}_\infty, C^\infty_\cdot, W_\infty, F_\infty)$$

Expansion of the asymptotically conditionally normal law

Adjoint of the symbol.

$$
\begin{aligned}
&E\left[\sigma(z,x,\partial_z,\partial_x)^*\left\{\phi(z;W_\infty,C_\infty)\middle|F_\infty = x\right]p^{F_\infty}(x)\right\} \\
= \;& \sum_j (-\partial_z)^{m_j}(-\partial_x)^{n_j}\left(E\left[c_j(z,x)\phi(z;W_\infty,C_\infty)\middle|F_\infty = x\right]p^{F_\infty}(x)\right)
\end{aligned}
$$

if a random symbol σ has the form

$$
\sigma(z,x,iu,iv) \;=\; \sum_j c_j(z,x)(iu)^{m_j}(iv)^{n_j}
$$

for random mesurable functions $c_j : - \times \mathbb{R}^d \times \mathbb{R}^{d_1} \to \mathbb{R}$, where $m_j \in \mathbb{Z}_+^d$ and $n_j \in \mathbb{Z}_+^{d_1}$.

Expansion of the asymptotically conditionally normal law

Approximation to the (local-global or local-local) joint density of (Z_n, F_n):

$$
\begin{aligned}
p_n(z,x) &= E\left[\phi(z;W_\infty,C_\infty)\middle|F_\infty = x\right]p^{F_\infty}(x) \\
&\quad + r_n E\left[\sigma(z,\partial_z,\partial_x)^*\left\{\phi(z;W_\infty,C_\infty)\middle|F_\infty = x\right]p^{F_\infty}(x)\right\}.
\end{aligned}
$$

Expansion of the asymptotically conditionally normal law

Approximation to the (local) conditional density of $\mathcal{L}\{Z_n|F_\infty = x\}$:

$$
\begin{aligned}
p_n(z|x) &= E\left[\phi(z;W_\infty,C_\infty)\middle|F_\infty = x\right] \\
&\quad + r_n E\left[\sigma(z,\partial_z,\partial_x)^*\left\{\phi(z;W_\infty,C_\infty)\middle|F_\infty = x\right]p^{F_\infty}(x)\right\}/p^{F_\infty}(x).
\end{aligned}
$$

Set $F_n = F_\infty$ instead of assuming an expansion for F_n. Then $\tilde{F}_\infty(\omega, z) = 0$.

It is also possible to give an expansion of $\mathcal{L}\{Z_n|F_n\}$ with that of $\mathcal{L}\{F_n\}$.

Expansion of the asymptotically conditionally normal law

The random symbol is given in the talk.

Expansion of the asymptotically conditionally normal law

Under certain nondegeneracy conditions,

$$\left| E\left[f(Z_n, F_n) \right] - \int f(z,x) p_n(z,x)\, dz dx \right| \;=\; o(r_n)$$

$$\text{for } f \in \mathcal{E}(M,\gamma).$$

More precisely, the error bound involves terms like ones in Y (97).

Expansion of the asymptotically conditionally normal law

Under certain nondegeneracy conditions,

$$\left| E\left[f(Z_n) \middle| F_\infty = x \right] - \int f(z) p_n(z|x)\, dzdx \right| \;=\; o(r_n)$$

for $f \in \mathcal{E}(M, \gamma)$.

Non synchronous covariance estimator

- Asymptotic mixed normality for random diffusion coefficients under dependent sampling scheme (Hayashi and Y)

- Asymptotic expansion when the diffusion coefficients are deterministic (Dalalyan and Y)

Non synchronous covariance estimator

- Asymptotic expansion when the diffusion coefficients are random:

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asymptotic expansion of a nonsynchronous quadratic variation having a mixed normal limit