Hypotheses Testing: Poisson versus Self-Exciting and Self-Correcting

Sergueï DACHIAN
Université Blaise Pascal, Clermont-Ferrand
Yury A. KUTOYANTS
Université du Maine, Le Mans

Statement of the problem

Model: Stationary (counting) point process $X = \{X_t,\ t \geqslant 0\}$.

The simplest case — Poisson process with a constant intensity $S > 0$, i.e., the increments of X on the disjoint intervals are independent and distributed according to Poisson law

$$\mathbf{P}\big\{X_t - X_s = k\big\} = \frac{S^k\,(t-s)^k}{k!}\,\mathrm{e}^{-S(t-s)}, \quad 0 \leqslant s < t, \quad k = 0, 1, \ldots$$

Observations: Sequence $X^T = \{t_1, t_2, \ldots\}$ of events of X on $[0,T]$.

Purpose: Test if X is a Poisson process (with known intensity).

Alternatives: Self-exciting and self-correcting processes.

Asymptotics: $T \to \infty$.

Recalls

The process X admits a unique (Doob-Meyer) decomposition

$$X_t = \mathcal{A}_t + \mathcal{M}_t$$

where $\mathcal{M} = \{\mathcal{M}_t, \mathscr{F}_t, t \geqslant 0\}$ is a martingale and $\mathcal{A} = \{\mathcal{A}_t, \mathscr{F}_t, t \geqslant 0\}$ is predictable increasing process.

$$\mathcal{A}_t = \int_0^t S\left(v, \omega\right)\ \mathrm{d}v$$

where $\mathcal{S} = \{S\left(t, \omega\right), \mathscr{F}_t, t \geqslant 0\}$ is called *intensity function*.

Self-exciting processes

Introduced by **A.G. Hawkes** in 1972.

Defined by intensity function of the form

$$S(t, X) = S + \int_{-\infty}^{t} g(t - s)\ \mathrm{d}X_s$$

where $S > 0$ and the function $g(\cdot) \geqslant 0$ satisfies the condition

$$\rho = \int_{0}^{\infty} g(t)\ \mathrm{d}t < 1.$$

The *rate* of the process is $\mu = \dfrac{S}{1-\rho}$, $\quad \mathbf{E}X_t = \mu\, t.$

Example

Let $g\,(t) = \alpha\,\mathrm{e}^{-\gamma t}$, where $\alpha > 0, \gamma > 0$ and $\alpha/\gamma < 1$. Then the point process X with intensity function

$$S\,(t, X) = S + \alpha \sum_{t_i \leqslant t} \mathrm{e}^{-\gamma\,(t - t_i)}$$

is self-exciting with the rate

$$\mu = \frac{S\,\gamma}{\gamma - \alpha}\,.$$

One-sided parametric alternative

We assume that the observed process is either Poisson with constant intensity S_* or is self-exciting process with intensity function

$$S(t, X) = S_* + \theta \int_{-\infty}^{t} h(t-s)\ \mathrm{d}X_s, \qquad \theta > 0.$$

The function $h(\cdot)$ is supposed to be known and

$$h(\cdot) \in \mathscr{L}_+(\mathbb{R}_+) = \left\{ f(\cdot) \geqslant 0 \ : \ \int_0^{\infty} f(t)\ \mathrm{d}t < \infty \right\}.$$

To have contiguous alternatives we consider $\theta = \theta_T = u/\sqrt{T}$ and test

$$\mathscr{H}_0 : \quad u = 0 \qquad \text{against} \qquad \mathscr{H}_1 : \quad u > 0.$$

LAUMP tests

Randomised test: $\phi_T = \phi_T\left(X^T\right)$ is the probability to accept $\mathscr{H}_1$.

Size of the test: $\alpha(\phi_T) = \mathbf{E}_0\,\phi_T\left(X^T\right)$.

Power function of the test: $\beta_T\left(u,\phi_T\right) = \mathbf{E}_u\,\phi_T\left(X^T\right), \quad u \geqslant 0.$

For $\varepsilon \in (0,1)$ we denote $\mathscr{K}_\varepsilon = \left\{\phi_T : \lim\limits_{T\to\infty} \alpha(\phi_T) = \varepsilon\right\}$.

Definition. *A test ϕ_T^* is called locally asymptotically uniformly most powerful (LAUMP) in the class $\mathscr{K}_\varepsilon$ if for any other test $\phi_T \in \mathscr{K}_\varepsilon$ and any constant $K > 0$ we have*

$$\lim_{T\to\infty} \inf_{0\leqslant u\leqslant K} \left[\beta_T\left(u,\phi_T^*\right) - \beta_T\left(u,\phi_T\right)\right] \geqslant 0.$$

Test statistics

We consider

$$\Delta_T\left(X^T\right) = \frac{1}{S_*\sqrt{T\,\mathrm{I}_h^*}}\int_0^T\int_0^{t-} h\left(t-s\right)\,\mathrm{d}X_s\,\left[\mathrm{d}X_t - S_*\,\mathrm{d}t\right]$$

$$= \frac{1}{S^*\sqrt{T\,\mathrm{I}_h^*}}\sum_{0\leqslant t_j\leqslant T}\sum_{t_i<t_j} h\left(t_j-t_i\right) - \frac{1}{\sqrt{T\,\mathrm{I}_h^*}}\sum_{0\leqslant t_j\leqslant T}\int_0^{T-t_j} h\left(v\right)\,\mathrm{d}v$$

where

$$\mathrm{I}_h^* = \int_0^\infty h\left(t\right)^2\,\mathrm{d}t + S_*\left(\int_0^\infty h\left(t\right)\,\mathrm{d}t\right)^2.$$

First main result

Theorem. *Let* $h(\cdot) \in \mathscr{L}_+^1(\mathbb{R}_+)$ *and bounded. Then the test*

$$\hat{\phi}_T\left(X^T\right) = \mathbb{1}_{\left\{\Delta_T(X^T) > z_\varepsilon\right\}}$$

is LAUMP in the class $\mathscr{K}_\varepsilon$, *and for any* $u > 0$

$$\beta_T\left(u, \hat{\phi}_T\right) \longrightarrow \hat{\beta}(u) = \mathbf{P}\left\{\zeta > z_\varepsilon - u\sqrt{\mathrm{I}_h^*}\right\}.$$

Here and in the sequel we denote $\zeta \sim \mathscr{N}(0,1)$, and z_ε is the $1-\varepsilon$ quantile of ζ, i.e., $\mathbf{P}\{\zeta > z_\varepsilon\} = \varepsilon$.

Key elements of the proof

The likelihood ratio is LAN under hypothesis $\mathscr{H}_0$, because

$$Z_T(u) = \exp\left\{\int_0^T \ln\left(1 + \frac{u}{S_*\sqrt{T}}\int_0^{t-} h(t-s)\,\mathrm{d}X_s\right)\,\mathrm{d}X_t - \right.$$
$$\left. -\frac{u}{\sqrt{T}}\int_0^T\int_0^t h(t-s)\,\mathrm{d}X_s\,\mathrm{d}t\right\} =$$
$$= \exp\left\{u\sqrt{\mathrm{I}_h^*}\Delta_T\left(X^T\right) - \frac{u^2}{2}\,\mathrm{I}_h^* + r_T\left(u, X^T\right)\right\}$$

and $\mathscr{L}_0\left\{\Delta_T\left(X^T\right)\right\} \Longrightarrow \mathscr{N}(0,1)$.

Power of the test: the third lemma of Le Cam.

LAUMP: our test is asymptoticaly as good as N-P test for any simple alternative.

Simulations

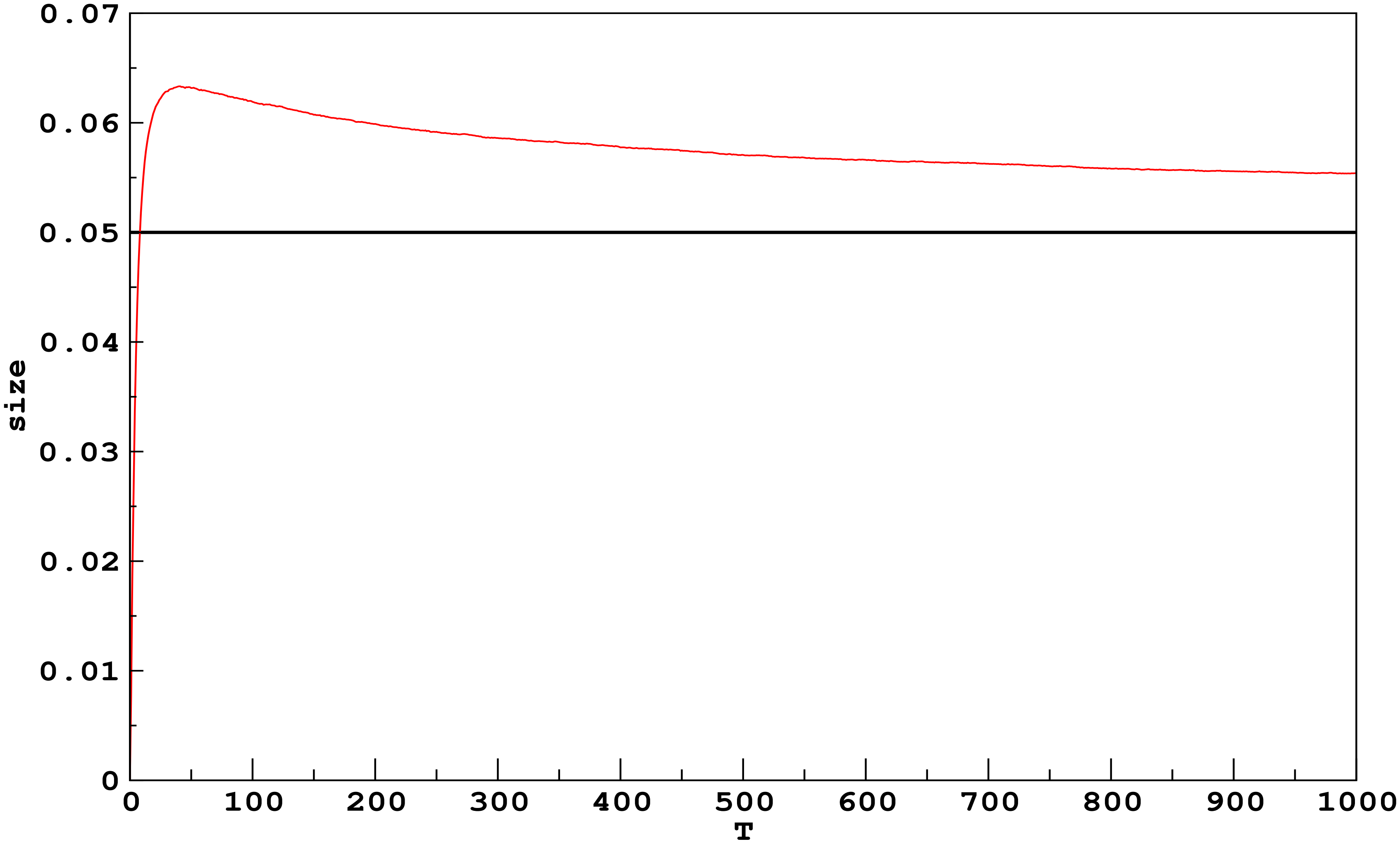
We take $S_* = 1$ and $h\left(t\right) = \dfrac{1}{2}\,\mathrm{e}^{-t/2}$, i.e.,

$$S\left(t, X\right) = 1 + \frac{u}{2\sqrt{T}} \sum_{t_i \leqslant t} \mathrm{e}^{-(t-t_i)/2}.$$

The test $\hat{\phi}_T^{\varepsilon} = \hat{\phi}_T\left(X^T\right) = \mathbb{1}_{\left\{\Delta_T(X^T) > z_\varepsilon\right\}}$ is LAUMP in $\mathscr{K}_\varepsilon$.

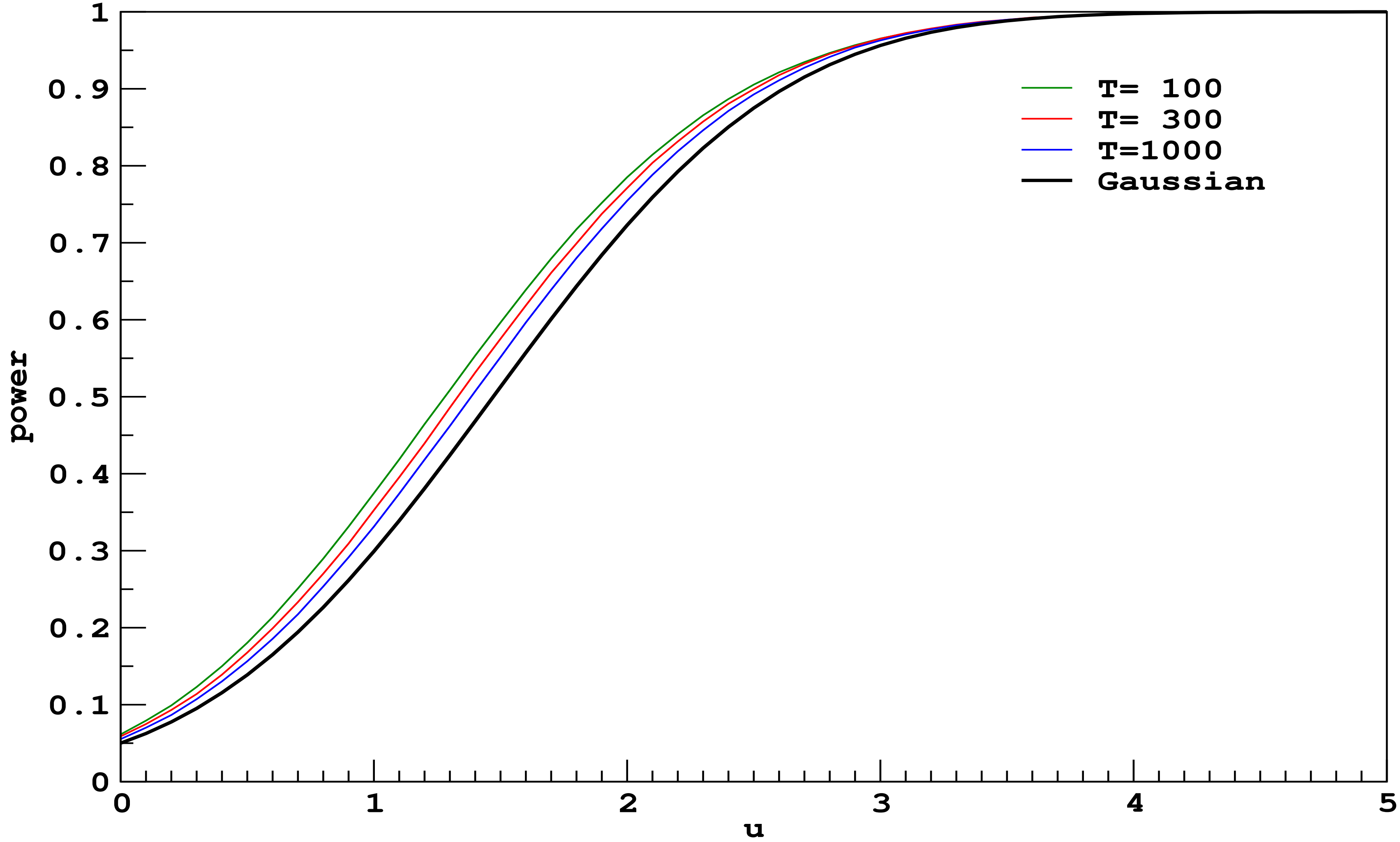
The test $\tilde{\phi}_T^{\varepsilon} = \tilde{\phi}_T\left(X^T\right) = \mathbb{1}_{\left\{\Delta_T(X^T) > z\right\}}$, where the threshold $z = z(\varepsilon, T)$ is chosen so that this test is of exact size ε, is also LAUMP.

Size of the test $\hat{\phi}_T^{0.05}$



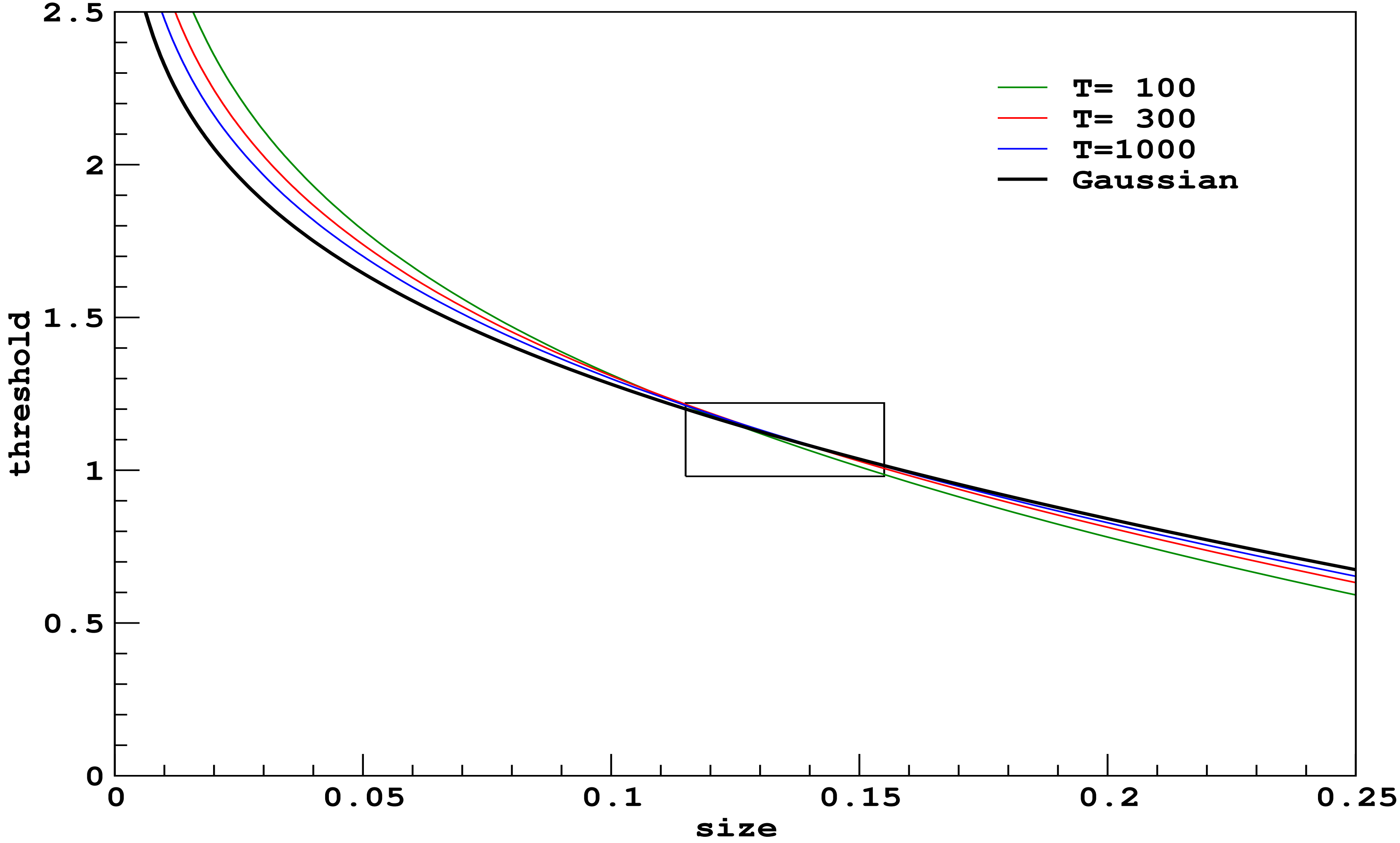
Obtained by simulating 10^7 Poisson trajectories

Power function of the test $\hat{\phi}_T^{0.05}$



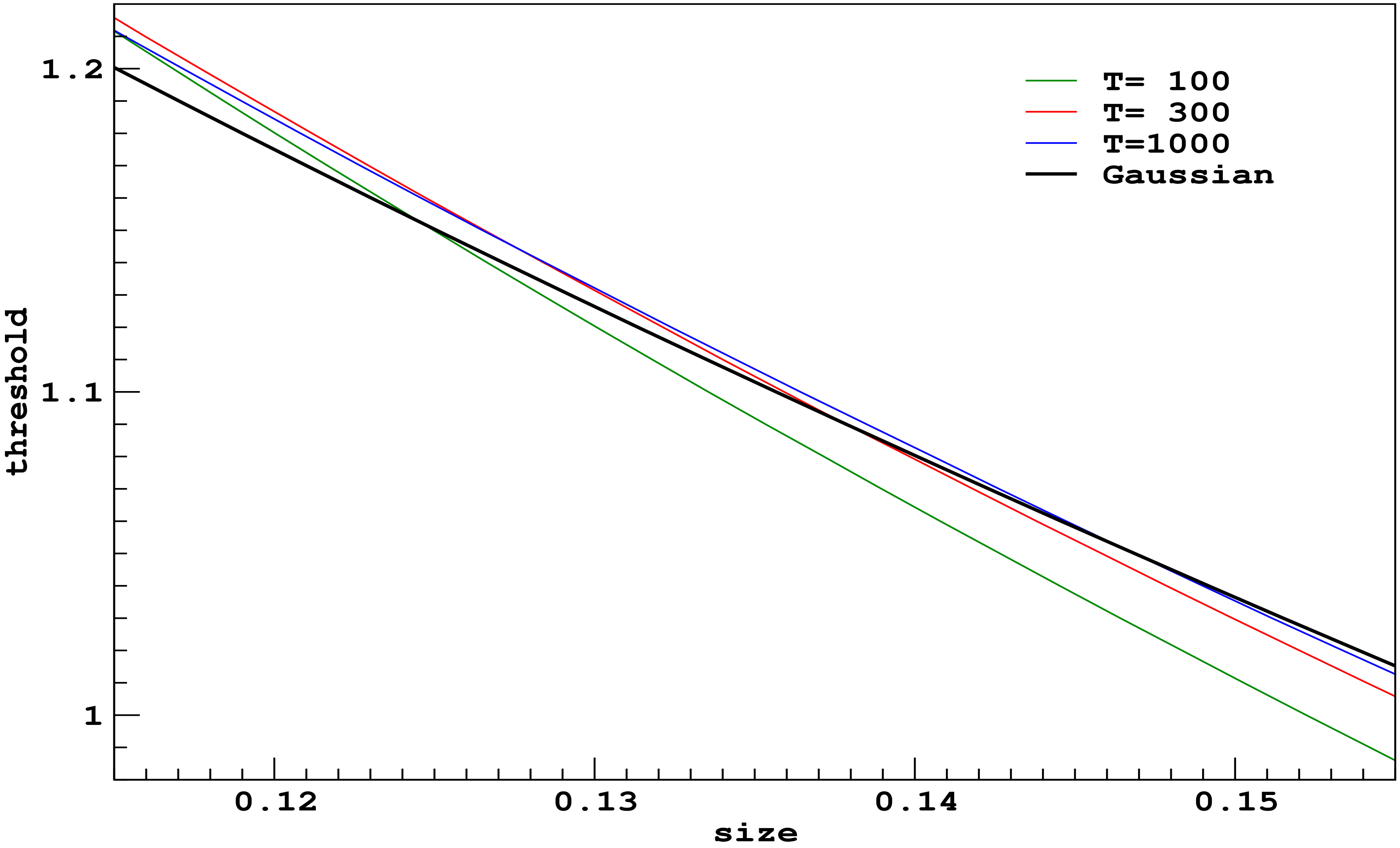
Obtained by simulating 10^6 self-exciting trajectories (for each u)

Threshold for the test $\tilde{\phi}_T^{0.05}$



$z \simeq 1.78$ for $T = 100$, $\quad 1.74$ for 300, $\quad 1.70$ for 1000, $\quad$ against $z_{0.05} \simeq 1.64$

Threshold for the test $\tilde{\phi}_T^{0.05}$ (zoom)



Obtained by simulating 10^7 Poisson trajectories

One-sided nonparametric alternative (1/3)

We assume that the observed process is either Poisson with constant intensity S_* or is self-exciting process with intensity function

$$S(t,X) = S_* + \frac{1}{\sqrt{T}} \int_{-\infty}^{t} u(t-s)\ \mathrm{d}X_s .$$

The function $u(\cdot)$ is now *unknown* but belongs to $\bigcup\limits_{r>0} \mathscr{U}_r$ with

$$\mathscr{U}_r = \left\{ u(\cdot) \in \mathcal{C}_+^b : \quad \int_0^\infty u(t)\ \mathrm{d}t = r,\ \operatorname{supp} u(\cdot)\ \text{is bounded} \right\}$$

where $\mathcal{C}_+^b$ is the set of nonnegative functions bounded by a same constant b.

One-sided nonparametric alternative (2/3)

We consider the following hypotheses testing problem

$$\mathscr{H}_0 : \quad u(\cdot) \equiv 0 \qquad \text{against} \qquad \mathscr{H}_1 : \quad u(\cdot) \in \mathscr{U}_r, \quad r > 0.$$

The power function $\beta_T(u, \phi_T) = \mathbf{E}_u \, \phi_T(X^T)$ of a test ϕ_T depends now on the function $u = u(\cdot) \in \mathscr{U}_r$ with some $r > 0$.

We seek to maximize the minimal power of test on the class $\mathscr{U}_r$.

However, for any test $\phi_T \in \mathscr{K}_\varepsilon$ we have

$$\lim_{T\to\infty} \inf_{u(\cdot)\in\mathscr{U}_r} \beta_T(u, \phi_T) = \varepsilon.$$

One-sided nonparametric alternative (3/3)

Hence we introduce the set

$$\mathscr{U}_{r,N} = \left\{ u\left(\cdot\right) \in \mathscr{U}_r : \quad \operatorname{supp} u\left(\cdot\right) \subset [0, N] \right\}$$

and denote

$$B_T\left(r, N, \phi_T\right) = \inf_{u(\cdot) \in \mathscr{U}_{r,N}} \beta_T\left(u, \phi_T\right).$$

Definition. *A test $\phi_T^*\left(\cdot\right)$ is called LAUMP in the class $\mathscr{K}_\varepsilon$ if for any other test $\phi_T\left(\cdot\right) \in \mathscr{K}_\varepsilon$ and any $K > 0$ we have*

$$\lim_{N\to\infty} \lim_{T\to\infty} \inf_{0 \leqslant r \leqslant K} \left[B_T\left(r, N, \phi_T^*\right) - B_T\left(r, N, \phi_T\right) \right] \geqslant 0.$$

Second main result

We consider the test

$$\hat{\phi}_T\left(X^T\right) = \mathbb{1}_{\left\{\Delta_T\left(X^T\right) > z_\varepsilon\right\}} \qquad \text{where} \qquad \Delta_T\left(X^T\right) = \frac{X_T - S_* T}{\sqrt{S_* T}}\ .$$

Theorem. *The test* $\hat{\phi}_T$ *is LAUMP in the class* $\mathscr{K}_\varepsilon$, *and for any* $u\left(\cdot\right) \in \mathscr{U}_r$

$$\beta_T\left(u, \hat{\phi}_T\right) \longrightarrow \hat{\beta}\left(u\right) = \mathbf{P}\left\{\zeta > z_\varepsilon - r\,\sqrt{S_*}\right\}\ .$$

Key elements of the proof

For any simple alternative $u\left(\cdot\right)$ the likelihood ratio can be written as

$$Z_T\left(u(\cdot)\right) = \exp\left\{\Delta_T\left(u, X^T\right) - \frac{1}{2}\,\mathrm{I}\left(u\right) + r_T\left(u, X^T\right)\right\}$$

with $\mathrm{I}\left(u\right) = \displaystyle\int_0^\infty u\left(t\right)^2\ \mathrm{d}t + S_*\ r^2.$

$\displaystyle\inf_{u(\cdot)\in\mathscr{U}_{r,N}} \mathrm{I}\left(u\right)$ is attained on $u^*\left(t\right) = \dfrac{r}{N}\,\mathbb{1}_{\{0\leqslant t\leqslant N\}}$ which corresponds to the *least favorable* for detection self-exciting process in $\mathscr{U}_{r,N}$.

Power of the test: the third lemma of Le Cam.

LAUMP: our test is “asymptoticaly” as good as N-P test for the least favorable alternative.

Simulations

We take $S_* = 1$ and $u(t) = \dfrac{r}{N}\mathbb{1}_{\{0 \leqslant t \leqslant N\}}$, i.e.,

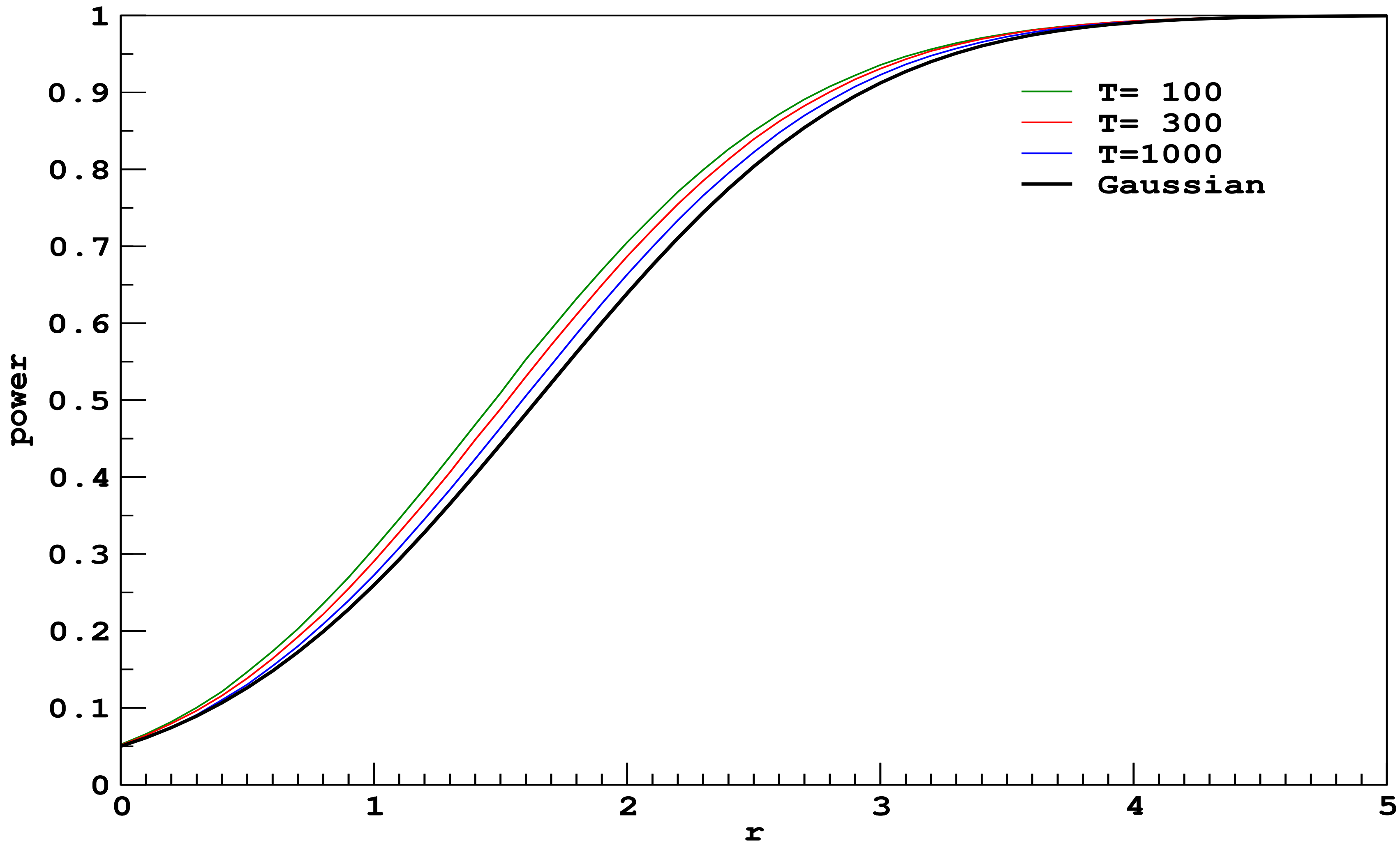
$$S(t, X) = 1 + \frac{r}{N\sqrt{T}} \sum_{t_i < t} \mathbb{1}_{\{t - t_i \leqslant N\}}.$$

This choice of $u(\cdot)$ allows us to compare the power function of our LAUMP test

$$\hat{\phi}_T^{\varepsilon}\left(X^T\right) = \mathbb{1}_{\left\{X_T > z_\varepsilon \sqrt{T} + T\right\}}$$

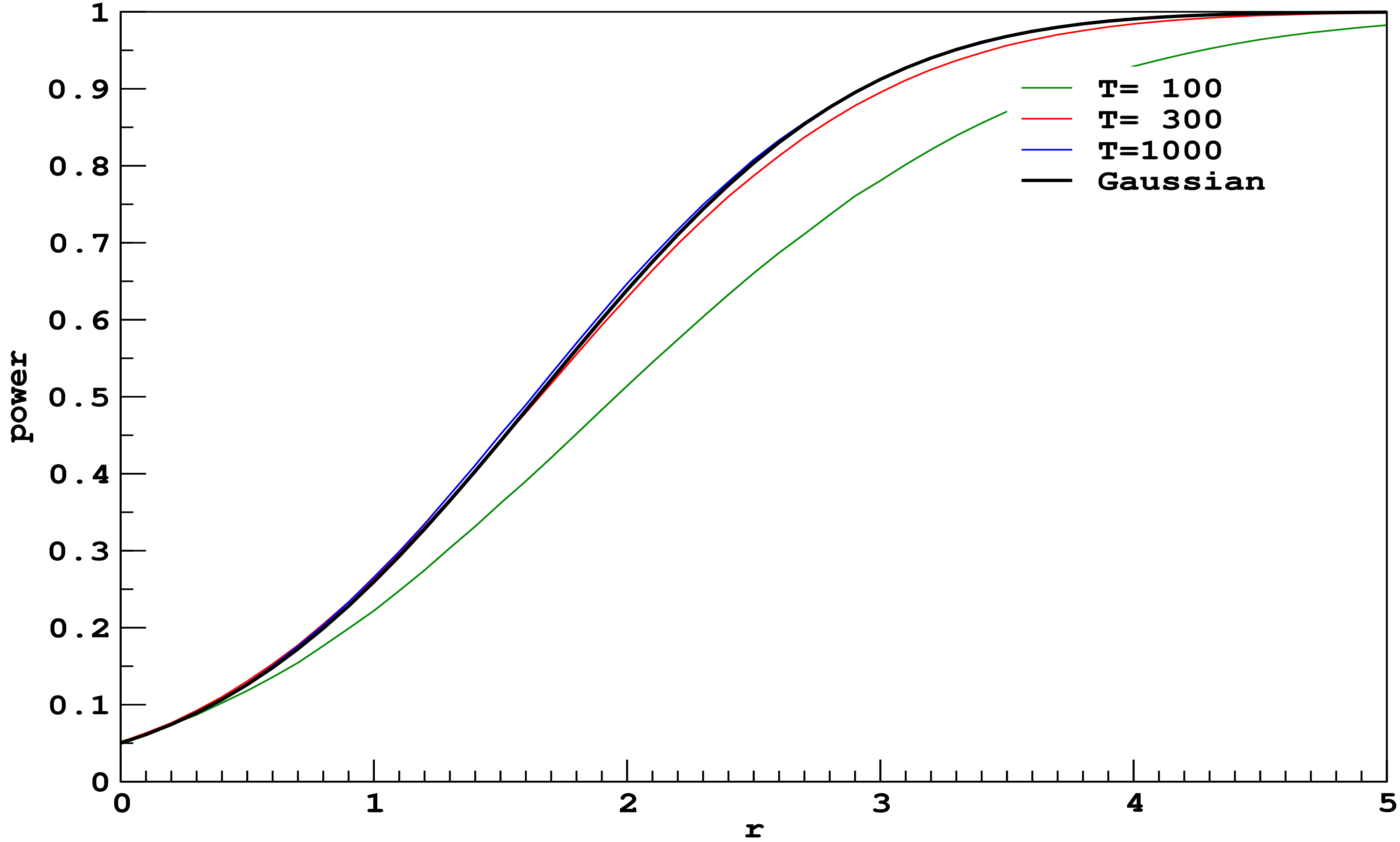
with the asymptotic power of Neyman-Pearson test for the least favorable alternative.

Power function of $\hat{\phi}_T^{0.05}$, $u(t) = \frac{r}{5}\mathbb{1}_{\{0\leqslant t\leqslant 5\}}$



Obtained by simulating 10^6 self-exciting trajectories (for each r)

Power function of $\hat{\phi}_T^{0.05}$, $u(t) = \frac{r}{50}\mathbb{1}_{\{0\leqslant t\leqslant 50\}}$



Good convergence is obtained if $1 \ll N \ll T$

Self-correcting processes

Introduced by **V. Isham** and **M. Westcott** in 1972.

Defined by intensity function of the form

$$S(t, X_t) = \Psi(at - X_t)$$

where $a > 0$ and the function $\Psi(\cdot)$ satisfies the following conditions:

- $$0 \leqslant \Psi(x) < \infty \text{ for any } x \in \mathbb{R},$$
- there exists some $c > 0$ such that $\Psi(x) \geqslant c$ for any $x > 0$,
- $$\varliminf_{x \to \infty} \Psi(x) > 1, \text{ and } \varlimsup_{x \to -\infty} \Psi(x) < 1.$$

Example

Let $b > 0$, $c > 0$ and

$$S\left(t, X_t\right) = \exp\left\{a + b\left(t - cX_t\right)\right\}.$$

Then the point process with such intensity function is self-correcting.

One-sided parametric alternative

We assume that the observed process is either Poisson with constant intensity S_* or is self-exciting process with intensity function

$$S\left(t, X_t\right) = S_*\, \psi\left(\theta\left[S_* t - X_t\right]\right), \qquad \theta > 0.$$

The function $\psi\left(\cdot\right)$ is supposed to be known, positive, continuously differentiable at 0, and such that $\psi\left(0\right) = 1$ and $\dot\psi\left(0\right) > 0$.

To have contiguous alternatives we consider $\theta = \theta_T = \dfrac{u}{S_*\, \dot\psi\left(0\right)\, T}$ and test

$$\mathscr{H}_0 : \quad u = 0 \qquad\qquad \text{against} \qquad\qquad \mathscr{H}_1 : \quad u > 0.$$

Test statistics

We consider

$$\Delta_T\left(X^T\right) = \frac{1}{S_* T}\int_0^T \left(S_* t - X_{t-}\right)\,\left[\mathrm{d}X_t - S_*\,\mathrm{d}t\right] =$$

$$= \frac{X_T - \left(X_T - S_* T\right)^2}{2 S_* T}\;.$$

Third main result

Theorem. *The test*

$$\hat{\phi}_T\left(X^T\right) = \mathbb{1}_{\{\Delta_T(X^T)>c_\varepsilon\}}$$

belongs to the class $\mathscr{K}_\varepsilon$ *and for any* $u > 0$ *its power function*

$$\beta_T(u,\hat{\phi}_T) \longrightarrow \beta(u,\hat{\phi}) = \mathbf{P}\left\{|\zeta| \leqslant h\left(u\right)\ z_{\frac{1-\varepsilon}{2}}\right\}.$$

Here

$$c_\varepsilon = \frac{1-z^2_{\frac{1-\varepsilon}{2}}}{2} \qquad \text{and} \qquad h\left(u\right) = \sqrt{\frac{2u}{1-e^{-2u}}}\,.$$

Key elements of the proof

The likelihood ratio is LAQ under hypothesis $\mathscr{H}_0$, because

$$Z_T(u) = \exp\left\{u\,\Delta_T\left(X^T\right) - \frac{u^2}{2}\,J_T(X^T) + r_T\left(u, X^T\right)\right\}$$

where $J_T\left(X^T\right) = \dfrac{1}{S_* T^2}\displaystyle\int_0^T \left(S_* t - X_t\right)^2 \, \mathrm{d}t.$

$$\mathscr{L}_0\left\{\Delta_T\left(X^T\right),\ J_T\left(X^T\right)\right\} \Longrightarrow \left(\frac{1 - W(1)^2}{2}\,,\quad \int_0^1 W\left(s\right)^2\, \mathrm{d}s\right)$$

where $\left\{W\left(s\right),\ 0 \leqslant s \leqslant 1\right\}$ is standard Wiener process.

Power of the test: the third lemma of Le Cam.

Simulations

We take $S_* = 1$ and $\psi(t) = e^t$, i.e.,

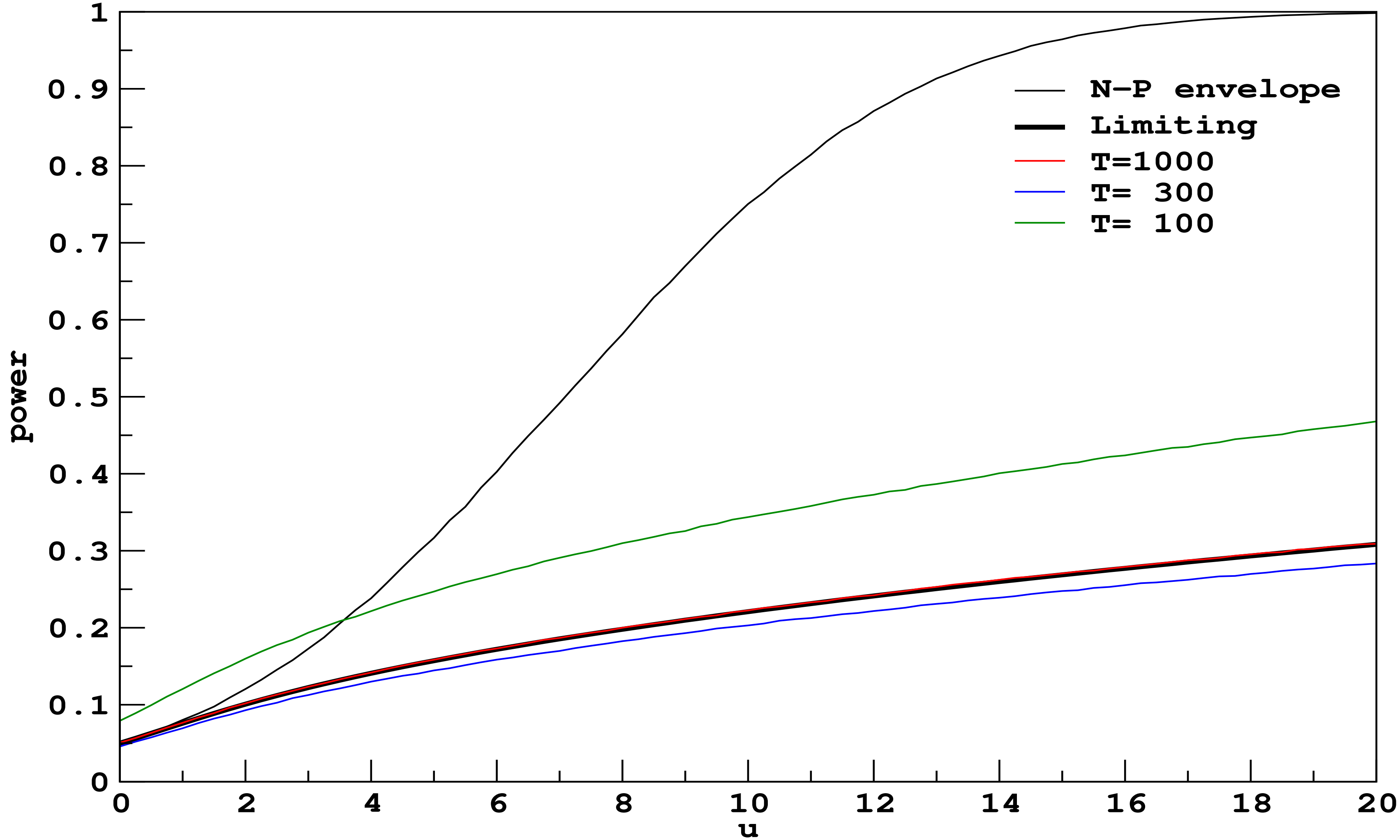
$$S(t, X_t) = e^{t - X_t}.$$

Simulations allows us to compare the power function of our test

$$\hat{\phi}_T\left(X^T\right) = \mathbb{1}_{\{\Delta_T(X^T) > c_\varepsilon\}}$$

with its asymptotic power, as well as with the Neyman-Pearson envelope (simulated for $T = 1000$).

Power function of $\hat{\phi}_T^{0.05}$



Obtained by simulating 10^6 self-correcting trajectories (for each u)

Remark

Note that these problems of hypotheses testing are similar to the corresponding problems of hypotheses testing for diffusion processes:

$$\mathrm{d}X_t = \theta_T \psi\left(X_t\right)\,\mathrm{d}t + \mathrm{d}W_t, \qquad X\left(0\right) = 0, \qquad 0 \leqslant t \leqslant T$$

where we put $\theta_T = u\,\varphi_T$ and test two hypotheses

$$\mathscr{H}_0 : \quad u = 0 \qquad\qquad \text{against} \qquad\qquad \mathscr{H}_1 : \quad u > 0,$$

i.e., Wiener process $\left(\mathscr{H}_0\right)$ against, say, ergodic diffusion $\left(\mathscr{H}_1\right)$. Then the statistic

$$\Delta_T^*\left(X^T\right) = \varphi_T \int_0^T \psi\left(X_t\right)\,\mathrm{d}X_t$$

has the similar to $\Delta_T\left(X^T\right)$ asymptotic behavior.

Remark (continuation)

For example, if $\psi(x) = -x$ (Ornstein-Uhlenbeck process under alternative), then the test

$$\phi^*\left(X^T\right) = \mathbb{1}_{\left\{\Delta_T^*\left(X^T\right) > c_\varepsilon\right\}}$$

with

$$\Delta_T^*\left(X^T\right) = -\frac{1}{T}\int_0^T X_t \,\mathrm{d}X_t \qquad \text{and} \qquad c_\varepsilon = \frac{1 - z^2_{\frac{1-\varepsilon}{2}}}{2}$$

has the same asymptotic properties as $\hat{\psi}\left(X^T\right)$.