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# ON INVARIANT DISTRIBUTION ESTIMATION FOR A CLASS OF CONTINUOUS-TIME STATIONARY PROCESSES

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I - Introduction
II - Asymptotic independence
III - Convergence of the empirical distribution function
IV - Asymptotic normality
V - Asymptotic variance estimation
VI - Weak convergence

# I - Introduction

$X := \{X_t,\, t \geq 0\}$

- continuous-time,
- real-valued
- stationary,
- $\mu$ invariant marginal distribution, law of $X_t$, $\forall t \geq 0$

$$F(x) := \mathrm{P}\big[X_0 \leq x\big]$$

$$\widehat{F}_T(x) := \frac{1}{T}\int_0^T \mathbb{I}_{\{X_s \leq x\}}\,\mathrm{d}s$$

*Example : diffusion process*

$$\mathrm{d}X_t = S(X_t)\mathrm{d}t + \mathrm{d}W_t$$

where $S : \mathbb{R} \longrightarrow \mathbb{R}$ Borelian with a polynomial majorant and

$$\limsup_{|x|\to\infty} \operatorname{sgn}(x)S(x) < 0.$$

Then

- existence and unicity of the invariant law $\mu$ with density $f(\,\cdot\,)$
- $\sqrt{T}\Big(\widehat{F}_T(x) - F(x)\Big) \xrightarrow{loi} \mathcal{N}\big(0, \sigma^2(x)\big)$

$$\sigma^2(x) = 2\int_0^\infty \Big(F_t(x,x) - F(x)^2\Big)\,\mathrm{d}t$$

$$= 4f(x)^2 \mathrm{E}\left[\left(\frac{F(\xi\wedge x) - F(\xi)F(x)}{f(\xi)}\right)^2\right]$$

$$\mathcal{L}(\xi) = \mu$$

– Estimation of the asymptotic variance

$$\widehat{\sigma}_x^2 := 4\widehat{f}(x)^2 \frac{1}{T} \int_0^T \left( \frac{\widehat{F}_T(X_t \wedge x) - \widehat{F}_T(X_t)\widehat{F}_T(x)}{\widehat{f}_T(X_t) + T^{-1/4}} \right)^2 \mathrm{d}t$$

Then (D. & Kutoyants 2004)

$$\widehat{\sigma}_x^2 \xrightarrow{\mathrm{P}} \sigma_x^2$$

– TLC fonctionnel

$$\left\{ \sqrt{T}\big(\widehat{F}_T(x) - F(x)\big) : x \in \mathbb{R} \right\} \xrightarrow{law} G, \quad \big(\mathcal{C}_0(\mathbb{R}), \|\cdot\|_\infty\big)$$

$G = \left\{ G(x) : x \in \mathbb{R} \right\}$ Gaussian process (Negri 1998).

*Discrete-time process*

1) $\big(Y_n\big)_n$ i.i.d. r.v. distribution $F(\,\cdot\,)$.

$$\widehat{F}_n(x) := \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\{Y_k \le x\}}.$$

Then

$$\sqrt{n}\Big(\widehat{F}_n(x) - F(x)\Big) \xrightarrow{loi} \mathcal{N}\big(0, \sigma^2(x)\big)$$

$$\sigma^2(x) = F(x)\big(1 - F(x)\big)$$

$$\widehat{\sigma}_n^2(x) := \widehat{F}_n(x)\big(1 - \widehat{F}_n(x)\big) \longrightarrow \sigma^2(x) \qquad \text{a.s.} \qquad \text{q.m.}$$

$$\Big\{\sqrt{n}\big(\widehat{F}_n(x) - F(x)\big) : x \in \mathbb{R}\Big\} \xrightarrow{law} G, \quad \big(\mathbb{D}(\mathbb{R}), \|\,\cdot\,\|_\infty\big)$$

$G$ Gaussian process (voir Dudley 1984).

2) $\big(Y_n\big)_n$ strongly mixing stationary process with

$$\sum_n \alpha_n < \infty$$

Then

$$\sqrt{n}\Big(\widehat{F}_n(x) - F(x)\Big) \xrightarrow{law} \mathcal{N}\big(0, \sigma^2(x)\big)$$

$$\sigma^2(x) = F(x)\big(1 - F(x)\big) + 2\sum_{k=1}^{\infty}\big(F_k(x,x) - F(x)^2\big)$$

Futhermore

If $\sum_n n^2 \varphi_n^{1/2} < \infty$ (Billingsley 1968) or

if $F(\,\cdot\,)$ is continuous and $\alpha_n = \mathcal{O}(n^{-a})$, $a > 1$ (Rio 2000)

then

$$\Big\{\sqrt{n}\big(\widehat{F}_n(x) - F(x)\big) : x \in \mathbb{R}\Big\} \xrightarrow{law} G, \quad \big(\mathbb{D}(\mathbb{R}), \|\cdot\|_\infty\big)$$

$G$ Gaussian process.

## II - Asymptotic independence

**(CL*)** *Castellana Leadbetter*(1986)

$$\Gamma_T(A) \longrightarrow \Gamma(A), \qquad \forall A \in \mathcal{B}(\mathbb{R}^2) \qquad \text{where}$$

$$\Gamma_T(A) := \int_0^T \Big(\mu_t^{(2)}(A) - \mu \otimes \mu(A)\Big)\, \mathrm{d}t.$$

**(SM)** *Strong mixing (α-mixing)*(Rosenblatt 1956)

$$\sigma^s(X) := \sigma\big\{X(u) : u \le s\big\} \quad \text{et} \quad \sigma_s(X) := \sigma\big\{X(u) : u \ge s\big\}$$

$$\alpha(t) := \sup\Big\{\big|\mathrm{P}[A \cap B] - \mathrm{P}[A]\,\mathrm{P}[B]\big| : A \in \sigma^s(X),\, B \in \sigma_{s+t}(X),\, s \ge 0\Big\}.$$

$X$ is *strongly mixing* when $\lim\limits_{t\to\infty} \alpha(t) = 0$.

Remark : $\qquad \alpha(\,\cdot\,) \in L^1([0,\infty)) \quad \Longrightarrow \quad (CL^*).$

*Examples*

1) Diffusion process (Veretennikov 1998)

$$\alpha(t) \le \exp\bigl(-\lambda t\bigr)$$

2) Exponentially ergodic Markov process

Markov process $X$ with

- $\bigl(P^t\bigr)_{t\ge 0}$ transition semi-group
- $\|P^t(x,\,\cdot\,) - \mu\|_{\mathrm{var}} \le M(x)\rho^t$
- $0 < \rho < 1$, $M \in L^1(\mu)$.

Then $X$ is strongly mixing and $\alpha(\,\cdot\,) \in L^1([0,\infty))$ with

$$\alpha(t) \le \int_{\mathbb{R}} M(x)\,\mu(\mathrm{d}x)\,\rho^t.$$

## III- Convergence of the empirical distribution function

$X$ stationary $\quad\Longrightarrow\quad \mathrm{E}\big[\widehat{F}_t(x)\big] = F(x) \qquad$ and

$$\operatorname{var}\widehat{F}_T(x) = \frac{2}{T}\int_0^T \Big(1-\frac{t}{T}\Big)\Big(F_t(x,x) - F(x)^2\Big)\,\mathrm{d}t$$

*Proposition 1* *(Asymptotic covariance)*
Under (CL*),

$$\begin{aligned} C(x,y) &:= \lim_{T\to\infty} T\operatorname{cov}[\widehat{F}_T(x), \widehat{F}_T(y)] \\ &= \int_0^\infty \Big(F_t(x,y) + F_t(y,x) - 2F(x)F(y)\Big)\,\mathrm{d}t. \end{aligned}$$

*(Asymptotic variance)*

$$\begin{aligned} \sigma^2(x) &:= \lim_{T\to\infty} T\operatorname{var}\big[\widehat{F}_t(x)\big] \\ &= 2\int_0^\infty \Big(F_t(x,x) - F(x)^2\Big)\,\mathrm{d}t = 2\,\Gamma\Big((-\infty,x]\times(-\infty,x]\Big). \end{aligned}$$

*Proposition 2* *(Almost-sure convergence)*

Assume that one of the two following conditions is satisfied

(i) $\alpha(t) = \mathcal{O}(t^{-\gamma})$ for some $0 < \gamma < 1$,

(ii) (CL$^*$).

Then

$$\lim_{T\to\infty} T^{\delta}\left|\widehat{F}_T(x) - F(x)\right| = 0 \qquad \text{a.s.}$$

for each $0 \leq \delta < \gamma/3$, in the case (ii) we take $\gamma = 1$.

The process is ergodic and satisfies *Glivenko-Cantelli* property

$$\mathrm{P}\left[\lim_{T\to\infty} \sup_x \left|\widehat{F}_T(x) - F(x)\right| = 0\right] = 1.$$

## IV - Asymptotic variance estimation

$$\sigma^2(x) = 2\int_0^\infty \Big(F_t(x,x) - F(x)^2\Big)\,\mathrm{d}t.$$

Let $0 < \eta < 1/3$ fixed

$$\widehat{F}_{t,T}(x) := \frac{1}{T}\int_0^T \mathbb{I}_{\{X_s \le x\}}\,\mathbb{I}_{\{X_{s+t}\le x\}}\,\mathrm{d}s$$

$$\widehat{\sigma}^2_T(x) := 2\int_0^{T^\eta} \Big(\widehat{F}_{t,T}(x) - \widehat{F}_T(x)^2\Big)\,\mathrm{d}t.$$

*Proposition 3* If $\alpha(\,\cdot\,) \in L^1([0,\infty))$ then $\lim\limits_{T\to\infty} \widehat{\sigma}^2_T(x) = \sigma^2(x)$ q.m.

*Proposition 4* Assume that $\alpha(t) = \mathcal{O}(t^{-\gamma})$ for some $\gamma > 1$.

Then $\lim\limits_{T\to\infty} T^\delta \left\{\widehat{\sigma}^2_T(x) - \sigma^2(x)\right\} = 0$

in quadratic mean for $0 < \eta < \frac{1}{3}$ and $\delta < \min\Big\{\frac{1}{2}(1-3\eta), \eta(\gamma-1)\Big\}$

almost-surely for $0 < \eta < \frac{1}{4}$, $\gamma > \frac{3}{2}$ and $\delta < \min\Big\{\frac{1}{3}(1-4\eta), \eta(\gamma-1)\Big\}$

Let

$$\widehat{C}_T(x,y) := \int_0^{T^\eta} \Big( \widehat{F}_{t,T}(x,y) + \widehat{F}_{t,T}(y,x) - 2\,\widehat{F}_T(x)\widehat{F}_T(y) \Big)\,\mathrm{d}t,$$

$$\widehat{F}_{t,T}(x,y) := \frac{1}{T}\int_0^T \mathbb{I}_{\{X_s \le x\}}\,\mathbb{I}_{\{X_{s+t}\le y\}}\,\mathrm{d}s$$

$0 < \eta < 1/3,$

then

$$\lim_{T\to\infty} \widehat{C}_T(x,y) = C(x,y) \quad \text{q.m.}$$

## V - Asymptotic normality

*Proposition 5*

If $\alpha(\,\cdot\,) \in \mathrm{L}^1([0,\infty))$ then

$$\widehat{G}_T(x) := \sqrt{T}\left(\widehat{F}_T(x) - F(x)\right) \xrightarrow{\text{law}} \mathcal{N}\big(0, \sigma^2(x)\big).$$

If in addition $\sigma(x) \neq 0$ we deduce that

$$\frac{\widehat{G}_T(x)}{\widehat{\sigma}^2_T(x)} := \frac{\sqrt{T}}{\widehat{\sigma}^2_T(x)}\left(\widehat{F}_T(x) - F(x)\right) \xrightarrow{\text{law}} \mathcal{N}\big(0, 1\big).$$

Sketch for the proof of Proposition 5

– $x$ fixed

– $\left|\sqrt{T}\{\widehat{F}_T(x) - F(x)\} - \sqrt{[T]}\{\widehat{F}_{[T]}(x) - F(x)\}\right| \leq 3\,[T]^{-1/2}$

– $Y_n(x) = \displaystyle\int_{n-1}^{n} \mathbb{I}_{\{X_t \leq x\}}\,\mathrm{d}t - F(x)$

– strongly-mixing stationary sequence of bounded zero-mean r.v.

– CLT (Ibraginov Linnik 1971)

– $\sqrt{N}\{\widehat{F}_N(x) - F(x)\} = \dfrac{1}{\sqrt{N}} \displaystyle\sum_{k=1}^{N} Y_k(x) \longrightarrow \mathcal{N}\big(0, \sigma^2(x)\big)$ in law

## VI - Weak convergence

Notations : (Pollard 1990, Rio 2000)

– $\mathbb{X} = \mathbb{R}$ with pseudo-metric

$$\rho(x, y) := |F(x) - F(y)|$$

– $\mathbb{B}(\mathbb{X})$ space of bounded measurable functions $f : \mathbb{X} \to \mathbb{R}$ endowed with the uniform norm $\|f(\,\cdot\,)\|_\infty = \sup_x |f(x)|$.

*Theorem 6* Assume that $\mathcal{O}(t^{-\gamma})$ for some $\gamma > 1$. Then

$$\widehat{G}_T = \Big\{ \sqrt{T}\big(\widehat{F}_T(x) - F(x)\big) : x \in \mathbb{R} \Big\} \xrightarrow{law} G, \quad \text{in} \quad \big(\mathbb{B}(\mathbb{X}),\, \|\,\cdot\,\|_\infty\big)$$

– $G := \{G(x) : x \in \mathbb{R}\}$ Gaussian process

– $\operatorname{cov}\big[G(x), G(y)\big] = C(x, y)$

– P-almost all the paths of $G$ are uniformly continuous from $\mathbb{X}$ into $\mathbb{R}$.

*Corollary 7*

$$\widehat{G}_T = \Big\{ \sqrt{T}\big(\widehat{F}_T(x) - F(x)\big) : x \in \mathbb{R} \Big\} \xrightarrow{law} G, \quad \big(\mathbb{D}(\mathbb{R}),\, \|\cdot\|_\infty\big)$$

*Corollaire 8*

If in addition $F(\,\cdot\,)$ is continuous, then

$$\widehat{G}_T = \Big\{ \sqrt{T}\big(\widehat{F}_T(x) - F(x)\big) : x \in \mathbb{R} \Big\} \xrightarrow{law} G, \quad \big(\mathcal{C}_b(\mathbb{R}),\, \|\cdot\|_\infty\big)$$

P-almost all paths of $G$ are uniformly continuous in $\mathbb{R}$.

Sketch for the proof of Theorem 6

i) $$\left|\sqrt{T}\{\widehat{F}_T(x) - F(x)\} - \sqrt{[T]}\{\widehat{F}_{[T]}(x) - F(x)\}\right| \leq 3\,[T]^{-1/2}$$

ii) (Rio 2000)

Theorem (Pollard 1990)

Let $(\mathbb{X}, \rho)$ be a totally-bounded pseudo-metric space, and let $\widehat{G}_n := \{\widehat{G}_n(x) : x \in \mathbb{X}\}$, $n \geq 0$ a sequence of measurable processes such that

– $\left(\widehat{G}_n(x_1), \ldots, \widehat{G}_n(x_k)\right)$ converges in law $\forall k, x_1, \ldots, x_k$

– $\forall \epsilon > 0, \forall \eta > 0, \exists \delta > 0,$

$$\limsup_{n\to\infty} \mathrm{P}^* \left[\sup_{\rho(x,y)<\delta} \left|\widehat{G}_n(x) - \widehat{G}_n(y)\right| > \eta\right] < \epsilon.$$

Then $\left(\widehat{G}_n\right)_n$ converges in law to a probability measure on $\mathbb{B}(\mathbb{X})$ concentrated on the subspace $\mathbb{U}(\mathbb{X}) \subset \mathbb{B}(\mathbb{X})$ of uniformly continuous bounded functions from $\mathbb{X}$ into $\mathbb{R}$.

– $G_N(A) := \sqrt{N}\Big\{\frac{1}{N}\int_0^N \mathbb{I}_{\{X_s\in A\}}\,\mathrm{d}s - \mu(A)\Big\} = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} Y_k(A)$

– $\pi_k(x) := F^{-1}\Big(2^{-k}\Big[2^k F(x)\Big]\Big)$

$$F(x) - 2^{-k} \le F\pi_k(x) \le F\pi_{k+1}(x) \le F(x)$$

– $\sup\limits_{\rho(x,y)\le\delta}\Big|\widehat{G}_N\big((y,x]\big)\Big| \le 2\sup\limits_{x\in\mathbb{R}}\Big|\widehat{G}_N\big(\pi_k(x),x\big]\Big| + \sup\limits_{\rho(x,y)\le\delta}\Big|\widehat{G}_N\big(\big(\pi_k(y),\pi_k(x)\big]\Big|$

– $\sup\limits_{x\in\mathbb{R}}\Big|\widehat{G}_N\big(\pi_k(x),x\big]\Big| \le \sum\limits_{l=k+1}^{n}\Delta_l + \Delta_k^* \qquad \text{where} \qquad 2^{n-1} < N \le 2^n$

– $\Delta_l = \max\limits_{j=1,\dots,2^l}\Big|\widehat{G}_N(I_{l,j})\Big| \qquad \text{where} \qquad I_{l,j} := \Big(F^{-1}\big((j-1)2^{-l}\big), F^{-1}\big(j2^{-l}\big)\Big\}^*$

– $\mathrm{E}\Delta_l = \int_0^\infty \mathrm{P}\big\{\Delta_l > z\big\}\,\mathrm{d}z \le c\,2^{-l/r^2}$ (by symmetrisation method)

$r = 4\gamma/(\gamma-1)$ $\square$

BIBLIOGRAPHY


Billingsley, P. (1968,1995), Wiley.

Castellana, J.V. and Leadbetter, M.R. (1986), *Stochastic Process. Appl.*

Davydov, Yu (2001), *Stat. Inference. Stoch. Process.*

Dehay, D. and Kutoyants, Yu.A. (2004), *J. Statist. Plann. Inference.*

Dehling, H., Mikosch, T. and Sørensen, M (2002), Birkhäuser.

Dudley, R.M. (1984), L.N.M. 1097, Springer.

Ibragimov I.A. and Linnik Yu.V. (1971), Wolters-Noordhoff.

Kutoyants, Yu.A. (2003), Springer.

Meyn, S.P. and Tweedie, R.L. (1993), *Adv. Appl. Prob.*

Negri, I. (1998), Inference Stoch. Process.

Pollard, D. (1990), I.M.S. and A.S.A.

Rio, E. (2000), Springer.

Shorack, G.R. and Wellner, J.A. (1986), Wiley.

Veretennikov, A.Yu. (1988), *Theory Probab. Appl.*

Veretennikov, A.Yu. (1999), *Stat. Inf. Stoch. Proc.*