On Two Problems of Hypotheses Testing for Poisson Processes

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Two Simple Hypotheses Testing For Poisson Process

(joint work with Yu. Kutoyants)

Realization $X^{(n)}$ of a Poisson process observed on the set $\mathbb{A}_n \subset \mathbb{R}^d$ is a random point measure, which on the set $\mathbb{B} \subset \mathbb{A}_n$ can be written as

$$X^{(n)}(\mathbb{B}) = \sum_{u_i \in \mathbb{A}_n} \chi_{\{u_i \in \mathbb{B}\}},$$

where $\{u_i\}$ are the *events* (random points) of the Poisson process, and $\chi_{\{\mathcal{D}\}}$ is the indicator function of the event $\mathcal{D}$.

The Poisson process with intensity function $S(u)$, $u \in \mathbb{A}_n$ is entirely defined by the following two conditions.

- for any collection of disjoint sets $\mathbb{B}_1, \ldots, \mathbb{B}_m \subseteq \mathbb{A}_n$ the random variables $X^{(n)}(\mathbb{B}_1), \ldots, X^{(n)}(\mathbb{B})$ are independent,
- for any $\mathbb{B} \in \mathbb{A}_n$ the random variable $X^{(n)}(\mathbb{B})$ has a Poisson distribution with the parameter $\Lambda_n(\mathbb{B}) = \int_{\mathbb{B}} S(u)\,\mathrm{d}u$.

We observe $X^{(n)}$ with the unknown intensity function $S(u),\ u \in \mathbb{A}_n$.
We want to test the following two simple hypotheses

$$\begin{aligned} \mathcal{H}_1 : \qquad & S(u) = S_1(u), \qquad && u \in \mathbb{A}_n, \\ \mathcal{H}_2 : \qquad & S(u) = S_2(u), \qquad && u \in \mathbb{A}_n, \end{aligned}$$

$S_1(\cdot)$, $S_2(\cdot)$ two known nonnegative functions.
We define the sets

$$\begin{aligned} \mathbb{B}_n &= \{u \in \mathbb{A}_n \ : \quad S_1(u) = 0\}\,, \\ \mathbb{C}_n &= \{u \in \mathbb{A}_n \ : \quad S_2(u) = 0\} \end{aligned}$$

Without of loss of generality: $\mathbb{B}_n \cap \mathbb{C}_n = \emptyset$
$\mathbf{P}_i^{(n)}$: the distribution of random element $X^{(n)}$ under $\mathcal{H}_i$, $i = 1, 2$.
$\mathbf{E}_1$ and $\mathbf{E}_2$ the corresponding mathematical expectations.

A test function ϕ :

$$\phi\left(X^{(n)}\right) = \text{Probability to accept } \mathcal{H}_2$$
$$\mathbf{E}_1\phi\left(X^{(n)}\right) = \text{The size of } \phi$$

For fixed $\varepsilon \in (0,1)$ we denote by $\mathcal{K}_\varepsilon$ the class of tests of the size ε (or level $1-\varepsilon$), i.e.

$$\mathcal{K}_\varepsilon = \left\{\phi : \quad \mathbf{E}_1\phi\left(X^{(n)}\right) = \varepsilon\right\}.$$

The power of ϕ is given by

$$\beta\left(\phi\right) = \mathbf{E}_2\phi\left(X^{(n)}\right).$$

We want to find the most powerful test in $\mathcal{K}_\varepsilon$.

$$\varepsilon < \exp\left\{-\int_{\mathbb{C}_n} S_1\left(u\right)\ \mathrm{d}u\right\} \equiv p_n,$$

because if $\varepsilon \geq p_n$, then it is sufficient to put

$$\phi\left(X^{(n)}\right) = \chi_{\left\{X^{(n)}(\mathbb{C}_n)=0\right\}}$$

with the power $\beta(\phi) = 1$.

Note that the quantity

$$L_n\left(X^{(n)}\right) = \int_{\mathbb{T}_n} \ln \frac{S_2(u)}{S_1(u)}\; X^{(n)}(\mathrm{d}u) - \int_{\mathbb{T}_n} \left[S_2(u) - S_1(u)\right]\; \mathrm{d}u$$

where the set

$$\mathbb{T}_n = \mathbb{A}_n \cap \mathbb{B}_n^c \cap \mathbb{C}_n^c$$

is well defined. The stochastic integral is the random sum

$$\int_{\mathbb{T}_n} \ln \frac{S_2(u)}{S_1(u)}\; X^{(n)}(\mathrm{d}u) = \sum_{u_i \in \mathbb{T}_n} \ln \frac{S_2(u_i)}{S_1(u_i)}.$$

L_n is the logarithm of the density of the absolutely continuous part of $\mathbf{P}_2^{(n)}$ w.r.t. $\mathbf{P}_1^{(n)}$.

Introduce the test

$$\hat{\phi}(X^{(n)}) = \begin{cases} 1, & \text{if } X^{(n)}(\mathbb{B}_n) > 0 \text{ or } \left(L_n\left(X^{(n)}\right) > c_\varepsilon,\, X^{(n)}(\mathbb{C}_n) = 0\right), \\ q_\varepsilon, & \text{if } L_n\left(X^{(n)}\right) = c_\varepsilon, \quad X^{(n)}(\mathbb{C}_n) = 0, \quad X^{(n)}(\mathbb{B}_n) = 0, \\ 0, & \text{if } X^{(n)}(\mathbb{C}_n) > 0 \text{ or } \left(L_n\left(X^{(n)}\right) < c_\varepsilon,\, X^{(n)}(\mathbb{B}_n) = 0\right), \end{cases}$$

where the numbers c_ε and q_ε satisfy the equation

$$\mathbf{E}_1 \hat{\phi}(X^{(n)}) = p_n\, \mathbf{P}_1^{(n)}(L_n\left(X^{(n)}\right) > c_\varepsilon) + $$
$$+ p_n\, q_\varepsilon\, \mathbf{P}_1^{(n)}(L_n\left(X^{(n)}\right) = c_\varepsilon) = \varepsilon.$$

According to Neyman-Pearson lemma we have the following proposition.

Theorem 1 *Suppose that the condition $\varepsilon < p_n$ is fulfilled, then the test $\hat{\phi}$ is the most powerful in the class $\mathcal{K}_\varepsilon$.*

Asymptotic Approach

we consider a sequence of problems of hypotheses testing
$\mathcal{H}_1, \mathcal{H}_2 = \left\{\mathcal{H}_1^{(n)}, \mathcal{H}_2^{(n)},\ n = 1, 2, \ldots\right\}$:

$$\mathcal{H}_1^{(n)} : \qquad S(u) = S_1(u), \qquad u \in \mathbb{A}_n,$$
$$\mathcal{H}_2^{(n)} : \qquad S(u) = S_2(u), \qquad u \in \mathbb{A}_n$$

where $\mathbb{A}_n \subset \mathbb{A}_{n+1} \nearrow \mathbb{A}$.
$\mathcal{K}'_\varepsilon$: the class of (sequences of) tests of asymptotic level $1 - \varepsilon$, i.e.

$$\mathcal{K}'_\varepsilon = \left\{\phi_n : \quad \lim_{n\to\infty} \mathbf{E}_1 \phi_n \left(X^{(n)}\right) = \varepsilon\right\}.$$

We want to construct a consistent test in the class $\mathcal{K}'_\varepsilon$.

We use the following notations :

$$Y_n = \int_{\mathbb{T}_n} l(u)\ X^{(n)}(\mathrm{d}u), \qquad l\,(u) = \ln \frac{S_2(u)}{S_1(u)},$$

$$\mu_{i,n} = \mathbf{E}_{_i} Y_n = \int_{\mathbb{T}_n} l\,(u)\ \ S_i(u)\,\mathrm{d}u, \qquad i = 1, 2,$$

$$\sigma^2_{i,n} = \mathbf{E}_i\left(Y_n - \mathbf{E}_{_i} Y_n\right)^2 = \int_{\mathbb{T}_n} l\,(u)^2\ \ S_i(u)\,\mathrm{d}u, \qquad i = 1, 2$$

$$\mathbb{T}_n\,(\delta) = \mathbb{T}_n \cap \left\{u : \quad |l\,(u)| > \delta\,\sigma_{1,n}\right\}, \quad \delta > 0.$$

Let us denote by z_ε the $1-\varepsilon$ quantile of the standard Gaussian law:

$$\mathbf{P}\left\{\zeta > z_\varepsilon\right\} = \varepsilon, \qquad \varepsilon \in \left(0,1\right), \quad \zeta \sim \mathcal{N}(0,1),$$

and put

$$\hat{p}_1 = \lim_{n\to\infty} \exp\left\{-\int_{\mathbb{C}_n} S_1\left(u\right)\,\mathrm{d}u\right\},$$

$$\hat{p}_2 = \lim_{n\to\infty} \exp\left\{-\int_{\mathbb{B}_n} S_2\left(u\right)\,\mathrm{d}u\right\}.$$

Of course $\hat{p}_1, \hat{p}_2 \in [0,1]$.

Theorem 2 *Let $\varepsilon \leq \hat{p}_1$, $\sigma_{1,n} \to \infty$ and for any $\delta > 0$ the condition*

$$\lim_{n\to\infty} \sigma_{1,n}^{-2} \int_{\mathbb{T}_n(\delta)} l\,(u)^2 \;\, S_1(u)\,\mathrm{d}u = 0$$

is fulfilled. Then the test

$$\hat{\phi}_n\left(X^{(n)}\right) = \max\left(\chi_{\left\{Y_n \geq c_{n,\varepsilon},\, X^{(n)}(\mathbb{C}_n)=0\right\}}, \chi_{\left\{X^{(n)}(\mathbb{B}_n)>0\right\}}\right)$$

with

$$c_{n,\varepsilon} = \sigma_{1,n}\; z_{\varepsilon/\hat{p}_1} + \mu_{1,n},$$

belongs to the class $\mathcal{K}'_\varepsilon$ and $\mathbf{E}_1\hat{\phi}_n\left(X^{(n)}\right) = \varepsilon + o\,(1)$. Moreover, if $\hat{p}_2 = 0$ or

$$\eta_n = \frac{\mu_{2,n} - \mu_{1,n}}{\sigma_{2,n}} - \frac{\sigma_{1,n}}{\sigma_{2,n}}\; z_{\varepsilon/\hat{p}_1} \longrightarrow \infty,$$

then the test $\hat{\phi}_n$ is consistent.

Edgeworth type expansion

The rate of approximation $o\,(1)$ can be improved using the Edgeworth type expansion of the distribution function of the statistic $\frac{Y_n-\mu_{1,n}}{\sigma_{1,n}}$ under hypothesis $\mathcal{H}_1$:

$$F_n(y) = \mathbf{P}_1^{(n)}\left\{\sigma_{1,n}^{-1}\int_{\mathbb{T}_n} l\,(u)\ \pi(\mathrm{d}u) < y\right\}.$$

Here $\pi\,(\mathrm{d}u) = X^{(n)}\,(\mathrm{d}u) - S_1(u)\,\mathrm{d}u$, is the centered Poisson process.

The expansion is obtained under the following two conditions:

$\mathcal{E}_1$. *There exists a sequence of real numbers* $\varepsilon_n \to 0$, *as* $n \to \infty$ *and constants* $C_r > 0, r = 3, 4$, *such that*

$$\int_{\mathbb{T}_n} |l\,(u)|^r \; S_1(u)\,\mathrm{d}u \le C_r\; \sigma_{1,n}^r\; \varepsilon_n^{r-2},$$

for $r = 3, 4$.

$\mathcal{E}_2$. *The following inequality*

$$\inf_{\frac{c_0\,\varepsilon_n^{-1}}{2}<\lambda<\frac{\varepsilon_n^{-2}}{2}} \int_{\mathbb{T}_n} \sin^2\left(\frac{\lambda\, l\,(u)}{\sigma_{1,n}}\right)\; S_1(u)\,\mathrm{d}u \ge \gamma \ln \varepsilon_n^{-1}$$

holds for small values of $c_0 > 0$ *and* $\gamma \ge 3/2$.

Let us denote

$$\gamma_{3,n} = \int_{\mathbb{T}_n} l(u)^3 \; S_1(u) \, \mathrm{d}u.$$

Theorem 3 *Let $\varepsilon \leq \hat{p}_1$, $\sigma_{1,n} \to \infty$ and the conditions $\mathcal{E}_1$, $\mathcal{E}_2$ be fulfilled. Then for the test*

$$\tilde{\phi}_n\left(X^{(n)}\right) = \max\left(\chi_{\left\{Y_n \geq b_{n,\varepsilon},\, X^{(n)}(\mathbb{C}_n)=0\right\}}, \chi_{\left\{X^{(n)}(\mathbb{B}_n)>0\right\}}\right)$$

with

$$b_{n,\varepsilon} = \mu_{1,n} + \sigma_{1,n}\, z_{\varepsilon/\hat{p}_1} - \frac{\gamma_{3,n}}{6\,\sigma_{1,n}^2}\left(1 - z_{\varepsilon/\hat{p}_1}^2\right),$$

we have $\tilde{\phi}_n \in \mathcal{K}'_\varepsilon$ and

$$\mathbf{E}_1 \tilde{\phi}_n\left(X^{(n)}\right) = \varepsilon + O\left(\varepsilon_n^2\right).$$

Furthermore $\tilde{\phi}_n$ is consistent under the conditions of Theorem 2.

Asymptotics of Power

To study the convergence of the power $\beta(\hat{\phi}_n)$, $\beta(\tilde{\phi}_n) \to 1$ we use (as usual in such situations) the *large deviations principle* for stochastic integral Y_n.

Introduce *logarithmic moment generating function* of Y_n (under $\mathcal{H}_2$):

$$\Lambda_n(\lambda) = \ln \mathbf{E}_2 \exp(\lambda Y_n) = \int_{\mathbb{T}_n} \left[\left(\frac{S_2(u)}{S_1(u)} \right)^{\lambda} - 1 \right] S_2(u) \, \mathrm{d}u,$$

- the limits

$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{\Lambda_n(\lambda)}{\mu_{2,n}},$$
$$\gamma = \lim_{n\to\infty} \frac{\mu_{1,n} + z_{\varepsilon/\hat{p}_1}\sigma_{1,n}}{\mu_{2,n}}$$
$$\psi = \lim_{n\to\infty} \frac{\int_{\mathbb{B}_n} S_2(u)\,\mathrm{d}u}{\mu_{2,n}},$$

- the Fenchel-Legendre transform of $\Lambda(\cdot)$:

$$\Lambda^*(\nu) = \sup_{\lambda\in\mathbb{R}}\{\lambda\nu - \Lambda(\lambda)\},$$

- $D^\circ_{\Lambda^*}$ and D°_{Λ} are the interior of the sets

$$D_{\Lambda^*} = \{\nu : \quad \Lambda^*(\nu) < \infty\},$$
$$D_{\Lambda} = \{\lambda : \quad \Lambda(\lambda) < \infty\}.$$

The behavior of the power is described below in the asymptotic

$$\mu_{2,n} \longrightarrow \infty.$$

Hence $\mathbf{E}_2 Y_n = \mu_{2,n}$ plays the role of *natural parameter* like n in the i.i.d. case. Moreover, we suppose that $\psi < \infty$, because if $\psi = \infty$, then the power function can be written as

$$\beta_n\left(\hat{\phi}_n\right) = 1 - \exp\left\{-\int_{\mathbb{B}_n} S_2\left(u\right) \mathrm{d}u(1 + \circ(1))\right\}.$$

Theorem 4 *Let the following conditions be fulfilled:*

- *for all* λ *the limit* $\Lambda(\lambda)$ *exists, finite or infinite,*
- *the limit* $\gamma < 1$ *exists and* $\gamma \in D^{\circ}_{\Lambda^*}$,
- *the function* $\Lambda(\cdot)$ *is differentiable in* D°_{Λ} *and for any* $\nu < 1$ *in* $D^{\circ}_{\Lambda^*}$, *there exists* $\eta \in D^{\circ}_{\Lambda}$ *such that* $\Lambda'(\eta) = \nu$.

Then the power of the tests $\hat{\phi}_n$ *and* $\tilde{\phi}_n$ *admits the representation*

$$\beta_n\left(\hat{\phi}_n\right) = 1 - \exp\left\{-\mu_{2,n}(\psi + \Lambda^*(\gamma))(1 + \circ(1))\right\}.$$

Example 1. We observe the Poisson process $X^{(n)}$ on the circle

$$\mathbb{A}_n = \{u : \quad |u| \leq n\} \subset \mathbb{R}^2.$$

We have to test the following two hypotheses concerning the intensity function $S(\cdot)$ of the Poisson process

$$\begin{aligned}
&\mathcal{H}_1 : \qquad S(u) = S_1(u) \equiv e^{a\cos(\omega r)} \ \chi_{\{u \in \mathbb{B}_n^c\}} \qquad && u \in \mathbb{A}_n, \\
&\mathcal{H}_2 : \qquad S(u) = S_2(u) \equiv e^{b\cos(\omega r)} \qquad && u \in \mathbb{A}_n,
\end{aligned}$$

where $u = (x, y)$, $r = \sqrt{x^2 + y^2}$ the frequency $\omega > 0$ and a, b are two given constants such that $0 < a < b$. Under $\mathcal{H}_2$ the intensity function is positive, but under $\mathcal{H}_1$ it is zero on the set

$$\mathbb{B}_n = \{u : \quad x \leq 0, \ \ y \geq 0, \quad u \in \mathbb{A}_n\}$$

(the second quadrant of the circle $\mathbb{A}_n$).

The test

$$\tilde{\phi}_n\left(X^{(n)}\right) = \max\left(\chi_{\{Y_n \geq b_{n,\varepsilon}\}}, \chi_{\left\{X^{(n)}(\mathbb{B}_n)>0\right\}}\right)$$

is a consistent test in the class $\mathcal{K}'_\varepsilon$ and

$$\mathbf{E}_1\tilde{\phi}_n\left(X^{(n)}\right) = \varepsilon + O\left(\frac{1}{n^2}\right).$$

Here the constants

$$b_{n,\varepsilon} = c_* \, n \left[z_\varepsilon - \frac{d_*}{6\, c_*^3 \, n} \left(1 - z_\varepsilon^2\right) \right] + h_* n$$

$$d_* = \frac{3\omega\,(b-a)^3}{8} \int_0^{\frac{2\pi}{\omega}} \cos^3(\omega r)\, e^{a\cos(\omega r)}\, \mathrm{d}r$$

$$h_* = \frac{3\omega\,(b-a)}{8} \int_0^{\frac{2\pi}{\omega}} \cos(\omega r)\, e^{a\cos(\omega r)}\, \mathrm{d}r$$

$$c_*^2 = \frac{3\omega}{8}\,(b-a)^2 \int_0^{\frac{2\pi}{\omega}} \cos^2(\omega r) e^{a\cos(\omega r)}\, \mathrm{d}r.$$

For the power of $\tilde{\phi}_n$ (and $\hat{\phi}_n$) we have the representation

$$\beta_n\left(\tilde{\phi}_n\right) = 1 - \exp\left\{-c\,n^2\left(\psi + \Lambda^*(\gamma)\right)(1 + \circ(1))\right\},$$

where

$$\psi = \frac{\int_0^{\frac{2\pi}{\omega}} e^{b\cos(\omega r)}\,\mathrm{d}r}{3(b-a)\int_0^{\frac{2\pi}{\omega}} \cos(\omega r)\,e^{b\cos(\omega r)}\,\mathrm{d}r},$$

$$\Lambda^*(\gamma) = \frac{\int_0^{\frac{2\pi}{\omega}} \left[e^{b\cos(\omega r)} - e^{a\cos(\omega r)} - (b-a)\cos(\omega r)\,e^{a\cos(\omega r)}\right]\mathrm{d}r}{(b-a)\int_0^{\frac{2\pi}{\omega}} \cos(\omega r)\,e^{b\cos(\omega r)}\,\mathrm{d}r}$$

$$c = \frac{3\omega\,(b-a)}{8}\int_0^{\frac{2\pi}{\omega}} \cos(\omega r)e^{b\cos(\omega r)}\,\mathrm{d}r.$$

One Sided Alternative for Poisson Processes

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Let $X^{(n)}$ be a realization of a Poisson process with the intensity function $S(\vartheta, x), x \in \mathbb{A}_n \subseteq \mathbb{R}^d$. The parameter $\vartheta \in \Theta \subseteq \mathbb{R}^1$ is unknown and we have to test the following two hypotheses :

$$\mathcal{H}_0 : \quad \vartheta = \vartheta_0$$
$$\mathcal{H}_1 : \quad \vartheta > \vartheta_0$$

$\left\{\mathbf{P}_\vartheta^{(n)}, \vartheta \in \Theta\right\}$: the family of distributions of $X^{(n)}$.
$\mathcal{K}'_\varepsilon$: the class of sequence of tests of asymptotic level $1 - \varepsilon$

$$\mathcal{K}'_\varepsilon = \left\{\phi_n : \quad \lim_{n\to\infty} \mathbf{E}_{\vartheta_0} \phi_n\left(X^{(n)}\right) = \varepsilon\right\}.$$

The family of $\left\{\mathbf{P}_\vartheta^{(n)},\, \vartheta \in \Theta\right\}$ is *locally asymptotically normal* (LAN) at the point ϑ_0, if there exists a sequence $\varphi_n\left(\vartheta_0\right)$ such that for any $u \in U_n = \left\{u :\ \vartheta_0 + \varphi_n\left(\vartheta_0\right) u \in \Theta\right\}$, the likelihood ratio admits the representation

$$\frac{\mathrm{d}\mathbf{P}^{(n)}_{\vartheta_0+\varphi_n(\vartheta_0)u}}{\mathrm{d}\mathbf{P}^{(n)}_{\vartheta_0}}\left(X^{(n)}\right) = \exp\left\{u\,\Delta_n\left(\vartheta_0\right) - \frac{1}{2}u^2 + r_n\left(\vartheta_0, u, X^{(n)}\right)\right\}$$

where the random variable $\Delta_n\left(\vartheta_0\right) = \Delta_n\left(\vartheta_0, X^{(n)}\right)$ under $\mathbf{P}^{(n)}_{\vartheta_0}$ is asymptotically normal

$$\mathcal{L}_{\vartheta_0}\left\{\Delta_n\left(\vartheta_0\right)\right\} \Rightarrow \mathcal{N}\left(0,1\right)$$

and the random variable

$$\mathbf{P}^{(n)}_{\vartheta_0} - \lim_{n\to\infty} r_n\left(\vartheta_0, u, X^{(n)}\right) = 0.$$

Conditions of LAN at ϑ_0: A simplified form of conditions given by Yu. Kutoyants. Under these conditions we have LAN with

$$\varphi_n(\vartheta_0)^{-2} = \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)^2}{S\left(\vartheta_0, x\right)}\,\mathrm{d}x \to \infty,$$

where $\dot S(\vartheta, x)$ is the derivative with respect to ϑ and the stochastic integral

$$\Delta_n(\vartheta_0) = \varphi_n(\vartheta_0) \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0, x\right)}{S\left(\vartheta_0, x\right)}\,\pi(\mathrm{d}x)$$

where

$$\pi(\mathrm{d}x) = X^{(n)}(\mathrm{d}x) - S\left(\vartheta_0, x\right)\mathrm{d}x.$$

We define the *local alternative* as $\vartheta_u^{(n)} = \vartheta_0 + \varphi_n(\vartheta_0)u, \quad u > 0$

Denote the power of a test ϕ_n at $\vartheta_u^{(n)}$ by $\beta(u, \phi_n)$.

To compare the different tests we use the following definition:

Definition 1 *We call a test $\phi_n^* \in \mathcal{K}'_\varepsilon$ locally asymptotically uniformly most powerful (LAUMP) in the class $\mathcal{K}'_\varepsilon$ if for any other test $\phi_n \in \mathcal{K}'_\varepsilon$ we have :*

$$\lim_{n\to\infty} \inf_{0\le u\le K} \left[\beta_n\left(u, \phi_n^*\right) - \beta_n\left(u, \phi_n\right)\right] \ge 0,$$

for any $K > 0$.

Theorem 5 *Under the conditions of LAN at ϑ_0 the test $\bar{\phi}_n\left(X^{(n)}\right) = \chi_{\{\Delta_n(\vartheta_0) > z_\varepsilon\}}$ is LAUMP in the class $\mathcal{K}'_\varepsilon$ and its power function*

$$\beta_n\left(u, \bar{\phi}_n\right) = \mathbf{P}\left\{\zeta > z_\varepsilon - u\right\} + o\left(1\right),$$

where $\zeta \sim \mathcal{N}\left(0,1\right)$.

Remark 1 *If in addition, the conditions of the expansion for $\Delta_n(\vartheta_0)$ are satisfied, then the test $\phi_n^* = \chi_{\{\Delta_n(\vartheta_0)>c_n\}}$ is LAUMP with the same representation*

$$\beta_n\left(u,\phi_n^*\right) = \mathbf{P}\left\{\zeta > z_\varepsilon - u\right\} + o\left(1\right).$$

Here the threshold

$$c_n = z_\varepsilon - \frac{\gamma_{3,n}}{6}\left(1 - z_\varepsilon^2\right)$$

where

$$\gamma_{3,n} = \varphi_n\left(\vartheta_0\right)^3 \int_{\mathbb{A}_n} \frac{\dot S\left(\vartheta_0,x\right)^3}{S\left(\vartheta_0,x\right)^2}\,\mathrm{d}x.$$

The size of ϕ_n^* is given by

$$\mathbf{E}_{\vartheta_0}\phi_n^*\left(X^{(n)}\right) = \varepsilon + O\left(\varepsilon_n^2\right).$$

First and second order efficiency of ϕ_n^*

The simple hypotheses: $\mathcal{H}_0 : \vartheta = \vartheta_0, \quad \mathcal{H}_u^{(n)} : \vartheta = \vartheta_0 + \varphi_n(\vartheta_0)u.$

$\tilde{\phi}_n$: Most powerful test (Neyman-Pearson Lemma) with the power

$$\beta_n\left(u, \tilde{\phi}_n\right) = \mathbf{P}\left\{\zeta > z_\varepsilon - u\right\} + o\left(1\right).$$

Therefore ϕ_n^* is *first order efficient*, that is the difference

$$\beta_n(u, \tilde{\phi}_n) - \beta_n\left(u, \phi_n^*\right) = o\left(1\right).$$

Theorem 6 *Under the regularity conditions the test* $\phi_n^* = \chi_{\{\Delta_n(\vartheta_0) > c_n\}}$ *is second order efficient, i.e.,*

$$\beta_n(u, \tilde{\phi}_n) - \beta_n\left(u, \phi_n^*\right) = o\left(\varepsilon_n\right).$$

Example 2. Suppose that we observe a realization $X^{(n)}$ of a Poisson process on the set $\mathbb{A}_n = [\,0, n]$, $n = 1, 2, \cdots$ with the intensity function

$$S(\vartheta, x) = \vartheta\, S(x) + \lambda, \qquad \vartheta > 0$$

where λ is a known positive constant (dark-current) and $S(x)$ is a known, nonnegative, and periodic function with the period $\tau > 0$. We have two hypotheses

$$\mathcal{H}_0 : \vartheta = \vartheta_0, \quad \mathcal{H}_1 : \vartheta > \vartheta_0.$$

All the conditions are satisfied with $\varepsilon_n = \varphi_n(\vartheta_0) = \frac{1}{\sqrt{n}}$ and the local alternative has the form

$$\mathcal{H}_u :\ \vartheta_u = \vartheta_0 + \frac{u}{\sqrt{n}} \qquad u > 0.$$

For the test

$$\phi_n^*\left(X^{(n)}\right) = \chi_{\{\Delta_n(\vartheta_0)\geq c_n\}}$$

with the threshold

$$c_n = z_\varepsilon - \frac{1-z_\varepsilon^2}{6\,\tau\,\sqrt{n}}\int_0^\tau \frac{S(x)^3}{\left(\vartheta_0 S(x)+\lambda\right)^2}\,\mathrm{d}x$$

we have

$$\beta_n(u,\tilde\phi_n) - \beta_n\left(u,\phi_n^*\right) = o\left(\frac{1}{\sqrt{n}}\right).$$